

OXFORD IB DIPLOMA PROGRAMME



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MATHEMATICS HIGHER LEVEL

COURSE COMPANION

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Mathematics as the science of patterns

Answers

Skills check

1 a $\{1, 2, 3, 4, 5\}$ b $\{-4, -3, -2, -1, 0, 1\}$

c $\{1, 2, 3, 4, 5, 6\}$

2 a $3(x-4) - 2(x+7) = 0$

$3x - 12 - 2x - 14 = 0$

$x = 26$

b $3x - 2(2x+5) = 2$

$3x - 4x - 10 = 2$

$-x = 12$

$x = -12$

c $5x + 4 - 2(x+6) = x - (3x-2)$

$5x + 4 - 2x - 12 = x - 3x + 2$

$3x - 8 = -2x + 2$

$5x = 10$

$x = 2$

3 a $2(\sqrt{3}-2) + \sqrt{3}(1-\sqrt{3}) = 2\sqrt{3} - 4 + \sqrt{3} - 3$
 $= 3\sqrt{3} - 7$

b $\frac{3}{\sqrt{2}} + 5\sqrt{2} = \frac{3\sqrt{2}}{2} + 5\sqrt{2} = \frac{13}{2}\sqrt{2}$

c $\frac{(1+\sqrt{3})}{(1-\sqrt{3})} = \frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{1+\sqrt{3}+\sqrt{3}+3}{1-3}$
 $= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

4 a $\frac{1}{(x-2)} = \frac{-3}{(1-2x)}$
 $1-2x = -3(x-2)$
 $1-2x = -3x+6$
 $x = 5$

b $\frac{2x}{2x^2+1} = \frac{1}{x-1}$
 $2x(x-1) = 2x^2+1$
 $2x^2-2x = 2x^2+1$
 $-2x = 1$
 $x = -\frac{1}{2}$

5 a 35 b -10

Exercise 1A

1 a 0, 1.5, 3

b $\frac{9}{10}, \frac{11}{12}, \frac{13}{14}$

c $\frac{1}{99}, \frac{1}{143}, \frac{1}{195}$

(denominators can be written as $1 \times 3, 3 \times 5, 5 \times 7, 7 \times 9, 9 \times 11, 11 \times 13, 13 \times 15$)

2 a $r(r+1)$

b $\frac{1}{r^2+1}$

c $2r-3$

3 a 1, 5, 9, 13

b $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

c $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

4 a $2+6+12+20$

b $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11}$

c $-1+4-9+16-25$

5 a $\sum_{r=1}^{\infty} 4r-5$

b $\sum_{r=1}^{10} (-1)^r$

c $\sum_{r=1}^6 6(-2)^{r-1}$

Investigation - quadratic sequences

$N = n^2 - 2n + 3$

$n = p-1 \Rightarrow n^2 - 2n + 3 = (p-1)^2 - 2(p-1) + 3$
 $= p^2 - 2p + 1 - 2p + 2 + 3$
 $= p^2 - 4p + 6$

$n = p \Rightarrow n^2 - 2n + 3 = p^2 - 2p + 3$

$n = p+1 \Rightarrow n^2 - 2n + 3 = (p+1)^2 - 2(p+1) + 3$
 $= p^2 + 2p + 1 - 2p - 2 + 3$
 $= p^2 + 2$

first differences are $2p-3$ and $2p-1$ second difference $= (2p-1) - (2p-3) = 2$ (a constant)

$N = 2n^2 + 2n + 1$

$n = p-1 \Rightarrow 2n^2 + 2n + 1 = 2(p-1)^2 + 2(p-1) + 1$
 $= 2p^2 - 4p + 2 + 2p - 2 + 1$
 $= 2p^2 - 2p + 1$

$n = p \Rightarrow 2n^2 + 2n + 1 = 2p^2 + 2p + 1$

$n = p+1 \Rightarrow 2n^2 + 2n + 1 = 2(p+1)^2 + 2(p+1) + 1$
 $= 2p^2 + 4p + 2 + 2p + 2 + 1$
 $= 2p^2 + 6p + 5$

first differences are $4p$ and $4p+4$ second difference $= 4$ (a constant)

$$N = -n^2 + 3n - 4$$

$$\begin{aligned} n = p - 1 &\Rightarrow -n^2 + 3n - 4 = -(p-1)^2 + 3(p-1) - 4 \\ &= -p^2 + 2p - 1 + 3p - 3 - 4 \\ &= -p^2 + 5p - 8 \end{aligned}$$

$$n = p \Rightarrow -n^2 + 3n - 4 = -p^2 + 3p - 4$$

$$\begin{aligned} n = p + 1 &\Rightarrow -n^2 + 3n - 4 = -(p+1)^2 + 3(p+1) - 4 \\ &= -p^2 - 2p - 1 + 3p + 3 - 4 \\ &= -p^2 + p - 2 \end{aligned}$$

first differences are $-2p + 4$ and $-2p + 2$

$$\begin{aligned} \text{second difference} &= (-2p + 2) - (-2p + 4) \\ &= -2 \text{ (a constant)} \end{aligned}$$

Conjecture: For the quadratic $N = an^2 + bn + c$ the second difference is a constant and is equal to $2a$.

Proof:

$$\begin{aligned} n = p - 1 &\Rightarrow an^2 + 6n + c = a(p-1)^2 + b(p-1) + c \\ &= ap^2 - 2ap + a + bp - b + c \end{aligned}$$

$$n = p \Rightarrow an^2 + bn + c = ap^2 + bp + c$$

$$\begin{aligned} n = p + 1 &\Rightarrow an^2 + bn + c = a(p+1)^2 + b(p+1) + c \\ &= ap^2 + 2ap + a + bp + b + c \end{aligned}$$

first differences are $2ap - a + b$ and $2ap + a + b$

second difference = $2a$, which proves the conjecture.

Investigation - triangular numbers

Since the second difference is a constant (1) the triangle numbers can be generated by a quadratic

$$N = an^2 + bn + c \quad 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$N = \frac{1}{2}n^2 + bn + c$$

$$n = 1 \Rightarrow \frac{1}{2} + b + c = 1 \Rightarrow b + c = \frac{1}{2}$$

$$n = 2 \Rightarrow 2 + 2b + c = 3 \Rightarrow 2b + c = 1$$

$$\therefore b = \frac{1}{2}, c = 0$$

$$N = \frac{1}{2}n^2 + \frac{1}{2}n \quad \text{or} \quad N = \frac{1}{2}n(n+1)$$

Investigation - more number patterns

Square numbers: $N = n^2$

Pentagonal numbers: $N = \frac{n(3n-1)}{2}$

Hexagonal numbers: $N = n(2n-1)$

Heptagonal numbers: $N = \frac{n(5n-3)}{2}$

Polygonal numbers	N
triangle	$\frac{1}{2}n(n+1) = \frac{n}{2}(n+1)$
square	$n^2 = \frac{n}{2}(2n+0)$
pentagon	$\frac{n(3n-1)}{2} = \frac{n}{2}(3n-1)$
hexagon	$n(2n-1) = \frac{n}{2}(4n-2)$
heptagon	$\frac{n(5n-3)}{2} = \frac{n}{2}(5n-3)$

Conjecture: For a polygon with k sides the polygonal numbers are given by

$$N = \frac{n}{2}[(k-2)n - (k-4)]$$

Exercise 1B

1 a $u_n = 5 + (n-1)6$

$$u_n = 6n - 1$$

b $u_n = 10 + (n-1)(-7)$

$$u_n = -7n + 17$$

c $u_n = a + (n-1)2$

$$u_n = 2n + a - 2$$

2 a $u_{15} = 2 + 14d = 2 + 14 \times 9 = 128$

b $u_{12} = -1 + 11d = -1 + 11 \times \frac{5}{4} = \frac{51}{4}$

c $u_n = 3 + (n-1)4 = 4n - 1$

3 $a + 3d = 18 \Rightarrow a - 15 = 18 \Rightarrow a = 33$

$$u_n = 33 + (n-1)(-5) = 38 - 5n$$

4 $a + 3d = 0$ (1)

$$a + 13d = 40$$
 (2)

$$(2) - (1) \Rightarrow 10d = 40 \Rightarrow d = 4$$

$$\therefore a + 12 = 0 \text{ and } a = -12$$

5 Salary after 15 years = $u_{16} = a + 15d$

$$= 48000 + 15 \times 500$$

$$= \text{€ } 55\,500$$

$$\text{Need } n \times 500 = 24000$$

$$\Rightarrow n = 48 \text{ years}$$

Exercise 1C

1 a $u_1 = 6 \quad d = 13 \quad u_n = 110$

$$6 + (n-1)13 = 110$$

$$(n-1)13 = 104$$

$$n-1 = 8$$

$$n = 9$$

$$S_9 = \frac{9}{2}(6 + 110) = 522$$

b $u_1 = 52 \quad d = -11 \quad u_n = -25$

$$52 + (n-1)(-11) = -25$$

$$(n-1)(-11) = -77$$

$$n-1 = 7$$

$$n = 8$$

$$S_8 = \frac{8}{2}(52 - 25) = 108$$

c $u_1 = -78 \quad d = -4 \quad u_n = -142$

$$-78 + (n-1)(-4) = -142$$

$$(n-1)(-4) = -64$$

$$n-1 = 16$$

$$n = 17$$

$$S_{17} = \frac{17}{2}(-78 - 142) = -1870$$

2 a $\sum_{r=1}^{10} 5r + 7 = 12 + 17 + 22 + \dots + 57$

$$= \frac{10}{2}(12 + 57)$$

$$= 345$$

$$\begin{aligned} \text{b } \sum_{r=1}^{15} 5-3r &= 2-1-4 \dots -40 \\ &= \frac{15}{2}(2-40) \\ &= -285 \end{aligned}$$

$$\begin{aligned} \text{3 } u_1 &= 60 \quad u_{10} = -3 \quad n = 16 \\ 60 + 9d &= -3 \\ 9d &= -63 \\ d &= -7 \end{aligned}$$

$$S_{16} = \frac{16}{2}(2 \times 60 + 15 \times -7) = 120$$

$$\begin{aligned} \text{4 } S_5 &= 25 \quad u_4 = 8 \\ \text{Let the numbers be} \\ u-2d, u-d, u, u+d, u+2d \\ S_5 &= u-2d+u-d+u+u+d+u+2d \\ \therefore 5u &= 25 \\ u &= 5 \end{aligned}$$

$$\begin{aligned} u_4 &= 8 \Rightarrow u+d=8 \Rightarrow d=3 \\ \text{The numbers are } &-1, 2, 5, 8, 11 \end{aligned}$$

$$\begin{aligned} \text{5 } S_n &= n(2n+3) \\ S_1 &= 1(2+3) = 5 \quad \therefore u_1 = 5 \\ S_2 &= 2(4+3) = 14 \quad \therefore u_1 + u_2 = 14 \quad \therefore u_2 = 9 \\ \therefore d &= 4 \\ u_1 &= 5, \quad u_2 = 9, \quad u_3 = 13, \quad u_4 = 17 \end{aligned}$$

Exercise 1D

$$\begin{aligned} \text{1 a } u_1 &= 1 \quad r = 2 \quad u_6 = 2^5 = 32 \quad u_n = 2^{n-1} \\ \text{b } u_1 &= 9 \quad r = \frac{1}{3} \quad u_6 = 9\left(\frac{1}{3}\right)^5 = \frac{1}{27} \quad u_n = 9\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-3} \\ \text{c } u_1 &= x^3 \quad r = \frac{1}{x} \quad u_6 = x^3\left(\frac{1}{x}\right)^5 = \frac{1}{x^2} \quad u_n = x^3\left(\frac{1}{x}\right)^{n-1} = \left(\frac{1}{x}\right)^{n-4} \\ \text{2 a } r &= \frac{1}{2}, u_{10} = ar^9 = 48 \times \frac{1}{512} = \frac{3}{32} \\ \text{b } r &= -\frac{8}{9} \div \frac{16}{3} = \frac{8}{9} \times \frac{3}{16} = -\frac{1}{6}, \\ u_5 &= ar^4 = \frac{16}{3} \times \frac{1}{1296} = \frac{1}{3 \times 81} = \frac{1}{243} \\ \text{3 a } a &= 0.03, r = 2 \\ \Rightarrow 0.03 \times 2^{n-1} &= 1.92 \Rightarrow 2^{n-1} = 64 \Rightarrow n = 7 \\ \text{b } a &= 81, r = \frac{1}{3} \\ 81 \times \left(\frac{1}{3}\right)^{n-1} &= \frac{1}{81} \Rightarrow \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^8 \Rightarrow n = 9 \\ \text{4 } ar^2 &= 2 \quad (1) \\ ar^4 &= 18 \quad (2) \\ (2) \div (1) &\Rightarrow r^2 = 9 \Rightarrow r = \pm 3 \\ u_2 &= ar = \frac{2}{9} \times \pm 3 = \pm \frac{2}{3} \\ \text{5 } 16r^4 &= 9 \Rightarrow r^4 = \frac{9}{16} \Rightarrow r = \pm \frac{\sqrt{3}}{2} \\ \Rightarrow u_7 &= 16r^6 = 16 \times \frac{27}{64} = \frac{27}{4} \end{aligned}$$

$$\begin{aligned} \text{6 } r &= \frac{a+2}{a-4} = \frac{3a+1}{a+2} \Rightarrow a^2 + 4a + 4 = 3a^2 - 11a - 4 \\ &\Rightarrow 0 = 2a^2 - 15a - 8 \\ &= (2a+1)(a-8) \\ \Rightarrow a &= -\frac{1}{2} \text{ or } 8 \\ \text{Hence } r &= \frac{1-\frac{1}{2}}{-4-\frac{1}{2}} = -\frac{1}{3} \text{ or } r = \frac{10}{4} = \frac{5}{2} \end{aligned}$$

Exercise 1E

$$\begin{aligned} \text{1 a } S_6 &= \frac{2\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 3.9375 \text{ or } \frac{63}{16} \\ \text{b } S_8 &= \frac{2(1-(-1.5)^8)}{1-(-1.5)} = -19.7 \text{ (3sf)} \text{ or } \frac{-1261}{64} \\ \text{c } \text{Sum} &= 1 + \frac{\frac{1}{2}\left(1-\left(\frac{-1}{2}\right)^9\right)}{1-\left(\frac{-1}{2}\right)} = 1.33 \text{ (3sf)} \text{ or } \frac{683}{512} \\ \text{d } u_1 &= 0.1, r = 0.2 \\ \text{Sum} &= \frac{0.1(1-0.2^{15})}{1-0.2} = \frac{1}{8}(1-0.2^{15}) \\ &= \frac{1}{8}\left(1-\frac{1}{5^{15}}\right) \\ &= 0.125 \text{ (3sf)} \\ \text{2 a } \sum_{r=0}^5 5^{3-r} &= 5^3 + 5^2 + 5^1 + 5^0 + 5^{-1} + 5^{-2} \\ &= \frac{125\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}} \\ &= 156.24 \text{ or } \frac{3906}{25} \\ \text{b } \sum_{r=0}^{n-1} 9 \times 10^r &= 9 + 9 \times 10 + 9 \times 10^2 + \dots + 9 \times 10^{n-1} \\ &= \frac{9(1-10^n)}{1-10} \\ &= 10^n - 1 \\ \text{3 } u_3 &= 2 \quad u_7 = \frac{1}{128} \\ u_1 r^2 &= 2 \quad u_1 r^6 = \frac{1}{128} \\ \frac{u_1 r^6}{u_1 r^2} &= \frac{\frac{1}{128}}{2} \\ \therefore r^4 &= \frac{1}{256} \\ r &= \frac{1}{4} \text{ or } -\frac{1}{4} \quad u_1 = 32 \\ S_6 &= \frac{32\left(1-\left(\frac{1}{4}\right)^6\right)}{1-\frac{1}{4}} = \frac{1365}{32} = 42.7 \\ \text{or } S_6 &= \frac{32\left(1-\left(-\frac{1}{4}\right)^6\right)}{1-\left(-\frac{1}{4}\right)} = \frac{819}{32} = 5.6 \end{aligned}$$

$$4 \quad a \quad u_1 = S_1 = \frac{3}{2} - 1 = \frac{1}{2}, u_2 = S_2 - S_1 = \left(\frac{3}{2}\right)^2 - \frac{3}{2} = \frac{3}{4},$$

$$u_3 = S_3 - S_2 = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 = \frac{9}{8}$$

$$b \quad u_n = \left(\frac{3}{2}\right)^n - \left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{n-1} \left(\frac{3}{2} - 1\right) \\ = \frac{1}{2} \times \left(\frac{3}{2}\right)^{n-1}$$

This is a GP with $u_1 = \frac{1}{2}$ and $r = \frac{3}{2}$

$$5 \quad P_n = a \times ar \times ar^2 \times \dots \times ar^{n-1} \\ = a^n r^{1+2+\dots+n-1} \\ = a^n r^{\frac{(n-1)n}{2}}$$

Reciprocal sequence = $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \dots, \frac{1}{ar^{n-1}}, \dots$

i.e. a GP with $u_1 = \frac{1}{a}$, and common ratio $\frac{1}{r}$.

$$R_n = \frac{1 \left(1 - \frac{1}{r^n}\right)}{1 - \frac{1}{r}} = \frac{1}{a} \times \frac{r^n - 1}{r^n} \times \frac{r}{r-1} = \frac{1}{a} \frac{r^n - 1}{(r-1)r^{n-1}}$$

$$\frac{S_n}{R_n} = \frac{a(1-r^n)}{1-r} \times \frac{a(r-1)r^{n-1}}{r^n - 1} = a \times -1 \times a \times -1 r^{n-1} \\ = a^2 r^{n-1}$$

$$\text{Hence } \left(\frac{S_n}{R_n}\right)^n = a^{2n} r^{n(n-1)} \\ = \left(a^n r^{\frac{(n-1)n}{2}}\right)^2 \\ = P_n^2 \quad \text{QED}$$

$$6 \quad ar = 24$$

$$ar^2 = 12(P-1) \Rightarrow r = \frac{P-1}{2}$$

$$\text{But } |r| < 1 \text{ so } -1 < \frac{P-1}{2} < 1 \text{ i.e. } -2 < P-1 < 2 \\ \Rightarrow -1 < P < 3 \quad (1)$$

$$\text{Also } S_3 = 76 \text{ so } \frac{48}{P-1} + 24 + 12(P-1) = 76$$

$$\Rightarrow 48 + 24(P-1) + 12(P-1)^2 = 76(P-1)$$

$$\Rightarrow 48 - 24 + 12P^2 + 12 = 76P - 76$$

$$\Rightarrow 12P^2 - 76P + 112 = 0$$

$$\Rightarrow 3P^2 - 19P + 28 = 0$$

$$(3P-7)(P-4) = 0$$

$$\Rightarrow P = \frac{7}{3} \text{ or } 4$$

From convergence condition (1), $P = \frac{7}{3}$

$$\text{Hence } r = \frac{\frac{4}{3}}{\frac{7}{3}} = \frac{4}{7}$$

$$7 \quad \text{The lengths are } a, ar, ar^2,$$

$$\text{Where } a + ar + ar^2 = 2 \quad (1)$$

$$\text{But } ar^2 = 2a$$

$$\text{so } r^2 = 2 \text{ and } r = \pm\sqrt{2}.$$

As a, ar, ar^2 are lengths, r must be positive so $r = \sqrt{2}$.

$$\text{Substitute into (1)} \Rightarrow a(1 + \sqrt{2} + 2) = 2$$

$$\Rightarrow a = \frac{2}{3+\sqrt{2}} = \frac{2}{7}(3-\sqrt{2}) \text{ metres.}$$

$$8 \quad 1, \frac{x+1}{3}, \frac{(x+1)^2}{9}, \frac{(x+1)^3}{27}$$

Convergent when $x = -1.5 = -\frac{3}{2}$

$$S_4 = 1 \frac{1 - \frac{(x+1)^4}{3^4}}{1 - \frac{x+1}{3}}$$

$$= \frac{185}{216}$$

$$9 \quad \frac{1}{1-r} = \frac{(1-r^n)}{1-r} = k r^{n-1}$$

$$\Rightarrow 1 - (1-r^n) = k r^{n-1} (1-r)$$

$$\Rightarrow r^n = k r^{n-1} (1-r)$$

$$\Rightarrow r = k(1-r)$$

$$\Rightarrow (1+k)r = k \Rightarrow r = \frac{k}{1+k}$$

$$\text{Hence } S = \frac{a}{1 - \frac{k}{1+k}} = \frac{a(1+k)}{1+k-k}$$

$$= a(1+k) = (k+1)a$$

$$= (k+1)u_1$$

Exercise 1F

$$1 \quad a \quad S = 4 u_2 \frac{u_1}{1-r} = 4 u_1 r$$

$$1 = 4r(1-r)$$

$$1 = 4r - 4r^2$$

$$4r^2 - 4r + 1 = 0$$

$$(2r-1)^2 = 0$$

$$r = \frac{1}{2}$$

$$b \quad u_1 = 32 \quad r = \frac{1}{2} \quad S_5 = \frac{32 \left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}} = 62$$

$$S = \frac{32}{1 - \frac{1}{2}} = 64$$

$$\text{percentage error} = \frac{2}{62} \times 100 = 3.23\%$$

$$2 \quad r = 1.5 \quad S_5 = 52750$$

$$\frac{u_1(1-1.5^5)}{1-1.5} = 52750$$

$$u_1 = \$4000$$

$$3 \quad a \quad 2 + 4 + 8 + 16 + 32 = 62$$

$$b \quad S_n > 1000000$$

$$\frac{2(1-2^n)}{1-2} > 1000000$$

$$(2^n - 1) > 500000$$

$$2^n > 500001$$

$$n = 19$$

- 4 a Let x = monthly repayment

$$\begin{aligned}\text{Amount owing after 1 month} \\ &= 1000 \times 1.01 - x\end{aligned}$$

$$\begin{aligned}\text{Amount owing after 2 months} \\ &= (1000 \times 1.01 - x) \times 1.01 - x \\ &= 1000 \times 1.01^2 - 1.01x - x\end{aligned}$$

$$\begin{aligned}\text{Amount owing after 3 months} \\ &= (1000 \times 1.01^2 - 1.01x - x) \times 1.01 - x \\ &= 1000 \times 1.01^3 - 1.01^2x - 1.01x - x\end{aligned}$$

$$\begin{aligned}\text{Amount owing after 24 months} \\ &= 1000 \times 1.01^{24} - 1.01^{23}x - 1.01^{22}x \\ &\quad - 1.01^{21}x \dots - 1.01x - x\end{aligned}$$

We require this to be zero

$$\begin{aligned}\therefore x + 1.01x + 1.01^2x + \dots + 1.01^{23}x \\ &= 1000 \times 1.01^{24}\end{aligned}$$

$$\begin{aligned}\frac{x(1-1.01^{24})}{1-1.01} &= 1000 \times 1.01^{24} \\ x &= \$47.07\end{aligned}$$

- b Total to be paid = 47.07×24
= \$1130

Exercise 1G

- 1 a Odd number + even number = $2a + 1 + 2b$
= $2(a + b) + 1$,

which is odd.

- b Odd number $\times$ odd number = $(2m + 1)(2n + 1)$
= $4mn + 2m + 2n + 1 = 2(m + n + 2mn) + 1$,
which is odd.

$$\begin{aligned}2 \quad \frac{1}{x-2} - \frac{2}{2x+5} &= \frac{2x+5-2(x-2)}{(x-2)(2x+5)} \\ \therefore \frac{1}{x-2} - \frac{2}{2x+5} &= \frac{9}{2x^2+x-10}\end{aligned}$$

$$\begin{aligned}3 \quad (a+b)^2 &= c^2 + 4\left(\frac{ab}{2}\right) \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ \therefore a^2 + b^2 &= c^2\end{aligned}$$

3	4	$3 \times 4 + 4$	16
7	8	$7 \times 8 + 8$	64
-6	-5	$-6 \times -5 + -5$	25
11	12	$11 \times 12 + 12$	144
8	9	$8 \times 9 + 9$	81

The product of two consecutive integers plus the larger of the two integers is equal to the square of the larger integer.

Proof: Let the two integers be n and $n + 1$

$$n(n+1) + (n+1) = (n+1)(n+1) = (n+1)^2$$

Exercise 1H

$$1 \quad p(n): S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\begin{aligned}\text{Step 1: when } n = 1, \text{ LHS} &= S_1 = u_1 \\ \text{RHS} &= \frac{u_1(1-r)}{1-r} = u_1\end{aligned}$$

$\therefore p(1)$ is true

$$\text{Step 2: assume } p(k) \text{ i.e., } S_k = \frac{u_1(1-r^k)}{1-r}$$

$$\text{Step 3: prove } p(k+1) \text{ i.e., } S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$$

$$\begin{aligned}\text{Proof: } S_{k+1} &= S_k + u_{k+1} \\ &= S_k + u_1 r^k \\ &= \frac{u_1(1-r^k)}{1-r} + u_1 r^k \\ &= \frac{u_1(1-r^k) + u_1 r^k(1-r)}{1-r} \\ &= \frac{u_1(1-r^k + r^k - r^{k+1})}{1-r} \\ \therefore S_{k+1} &= \frac{u_1(1-r^{k+1})}{1-r}\end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principle of mathematical induction, $p(n)$ is true

$$2 \quad a \quad p(n): \sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\begin{aligned}\text{Step 1: when } n = 1, \text{ LHS} &= 1 \\ \text{RHS} &= \frac{1}{6}(2)(3) = 1\end{aligned}$$

$\therefore p(1)$ is true

$$\text{Step 2: assume } p(k) \text{ i.e., } \sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$$

$$\text{Step 3: prove } p(k+1) \text{ i.e., } \sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k+2)(2k+3)$$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \\ &= \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)] \\ &= \frac{(k+1)}{6}[2k^2 + 7k + 6] \\ \therefore \sum_{r=1}^{k+1} r^2 &= \frac{(k+1)}{6}(k+2)(2k+3)\end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

$$b \quad p(n): \sum_{r=1}^n 2^{r-1} = 2^n - 1$$

$$\begin{aligned}\text{Step 1: when } n = 1, \text{ LHS} &= 2^0 = 1 \\ \text{RHS} &= 2^1 - 1 = 1\end{aligned}$$

$\therefore p(1)$ is true

$$\text{Step 2: assume } p(k) \text{ i.e., } \sum_{r=1}^k 2^{r-1} = 2^k - 1$$

Step 3: prove $p(k+1)$ i.e., $\sum_{r=1}^{k+1} 2^{r-1} = 2^{k+1} - 1$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} 2^{r-1} &= \sum_{r=1}^k 2^{r-1} + 2^k \\ &= 2^k - 1 + 2^k = 2(2^k) - 1 \\ \therefore \sum_{r=1}^{k+1} 2^{r-1} &= 2^{k+1} - 1\end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

c $p(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2}{4} (n+1)^2$

Step 1: when $n = 1$, LHS = $1^3 = 1$

$$\text{RHS} = \frac{1}{4} (2)^2 = 1$$

$\therefore p(1)$ is true

$$\begin{aligned}\text{Step 2: assume } p(k) \text{ i.e., } 1^3 + 2^3 + 3^3 + \dots + k^3 \\ = \frac{k^2}{4} (k+1)^2\end{aligned}$$

Step 3: prove $p(k+1)$ i.e.,

$$\begin{aligned}1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ = \frac{(k+1)^2}{4} (k+2)^2\end{aligned}$$

$$\begin{aligned}\text{Proof: } 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ = \frac{k^2}{4} (k+1)^2 + (k+1)^3 = \frac{(k+1)^2}{4} [k^2 + 4k + 4]\end{aligned}$$

$$\therefore 1^3 + 2^3 + 3^3 + \dots + (k+1)^3 = \frac{(k+1)^2}{4} (k+2)^2$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

d $p(n): \sum_{r=1}^n r(r+2) = \frac{n}{6} (n+1)(2n+7)$

Step 1: when $n = 1$, LHS = $1(3) = 3$

$$\text{RHS} = \frac{1}{6} (2)(9) = 3$$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$$\sum_{r=1}^k r(r+2) = \frac{k}{6} (k+1)(2k+7)$$

Step 3: prove $p(k+1)$ i.e.,

$$\sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)}{6} (k+2)(2k+9)$$

$$\begin{aligned}\text{Proof: } \sum_{r=1}^{k+1} r(r+2) &= \sum_{r=1}^k r(r+2) + (k+1)(k+3) \\ &= \frac{k}{6} (k+1)(2k+7) + (k+1)(k+3) \\ &= \frac{(k+1)}{6} [k(2k+7) + 6(k+3)] \\ &= \frac{(k+1)}{6} (2k^2 + 13k + 18) \\ \therefore \sum_{r=1}^{k+1} r(r+2) &= \frac{(k+1)}{6} (k+2)(2k+9)\end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

Exercise 1I

1 $p(n): 7^n - 1 = 6A \quad (A \in \mathbb{Z})$

Step 1: when $n = 1$, $7^n - 1 = 7 - 1 = 6$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$$7^k - 1 = 6A$$

Step 3: prove $p(k+1)$ i.e.,

$$7^{k+1} - 1 = 6B \quad (B \in \mathbb{Z})$$

Proof: $7^{k+1} - 1 = 7(7^k) - 1$

$$= 7(6A + 1) - 1$$

$$= 42A + 7 - 1$$

$$= 42A + 6$$

$$= 6(7A + 1)$$

$$\therefore 7^{k+1} - 1 = 6B$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

2 $p(n): 1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$

Step 1: when $n = 1$, LHS = 1

$$\text{RHS} = 1^2 = 1$$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$$1 + 3 + 5 + 7 + \dots + (2k-1) = k^2$$

Step 3: prove $p(k+1)$ i.e.,

$$\begin{aligned}1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1) \\ = (k+1)^2\end{aligned}$$

$$\begin{aligned}\text{Proof: } 1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1) \\ = k^2 + (2k+1)\end{aligned}$$

$$\therefore 1 + 3 + 5 + 7 + \dots + (2k+1) = (k+1)^2$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

3 $p(n): 9^n - 1 = 8A$, where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $9^n - 1 = 8$ $p(1)$ is true

Step 2: Assume $p(k)$ i.e.,

$$9^k - 1 = 8A$$

Step 3: prove $p(k+1)$ i.e.,

$$9^{k+1} - 1 = 8B \quad (B \in \mathbb{Z})$$

Proof: $9^{k+1} - 1 = 9 \times 9^k - 1$

$$= 9(8A + 1) - 1$$

$$= 72A + 8$$

$$= 8(9A + 1)$$

$$\therefore 9^{k+1} - 1 = 8B$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by induction, $p(n)$ is true

4 $p(n): n^3 - n = 6A$, where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $n^3 - n = 0 = 6 \times 0$

$\therefore p(1)$ is true

Step 2: Assume $p(k)$ i.e.

$$k^3 - k = 6A$$

Step 3: prove $p(k+1)$ i.e. $(k+1)^3 - (k+1) = 6B$
where $B \in \mathbb{Z}$

$$\begin{aligned} \text{Proof: } (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= 6A + k + 3k^2 + 2k \\ &= 6A + 3(k^2 + k) \\ &= 6A + 3k(k+1) \end{aligned}$$

But $k(k+1)$ is either odd $\times$ even or even $\times$ odd so is divisible by 2.

$\therefore 3k(k+1)$ is divisible by 6.

$$\therefore (k+1)^3 - (k+1) = 6B$$

$\therefore$ Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by induction $p(n)$ is true.

5 $p(n): \sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$

Step 1: when $n = 1$, $\sum_{r=1}^1 \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$

$$\text{and } \frac{n}{n+1} = \frac{1}{2} \therefore p(1) \text{ is true}$$

Step 2: assume $p(k)$ i.e. $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$

Step 3: prove $p(k+1)$ i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$

$$\begin{aligned} \text{Proof: } \sum_{r=1}^{k+1} \frac{1}{r(r+1)} &= \sum_{r=1}^k \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \end{aligned}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k+1)$ is true, by induction $p(n)$ is true.

6 $p(n): 2^{n+2} + 3^{2n+1} = 7A$ where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $2^{n+2} + 3^{2n+1} = 2^2 + 3^3$
 $= 8 + 27 = 35 = 7 \times 5$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e. $2^{k+2} + 3^{2k+1} = 7A$

Step 3: prove $p(k+1)$ i.e. $2^{k+3} + 3^{2k+3} = 7B$
where $B \in \mathbb{Z}$

$$\begin{aligned} \text{Proof: } 2^{k+3} + 3^{2k+3} &= 2(7A - 3^{2k+1}) + 3^{2k+3} \\ &= 14A + 3^{2k+3} - 2 \times 3^{2k+1} \\ &= 14A + 3^{2k+1}(9 - 2) \\ &= 14A + 3^{2k+1} \times 7 \\ &= 7(2A + 3^{2k+1}) = 7B \end{aligned}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k+1)$ is true, by induction, $p(n)$ is true.

7 $1, \frac{1}{3}, -\frac{1}{9}, -\frac{11}{27}, -\frac{49}{81}$

$$p(n): u_n = 3\left(\frac{2}{3}\right)^n - 1$$

Step 1: When $n = 1$ $u_1 = 1$ and $3\left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$
 $\therefore p(1)$ is true

Step 2: Assume $p(k)$ i.e. $u_k = 3\left(\frac{2}{3}\right)^k - 1$

Step 3: Prove $p(k+1)$ i.e. $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$

$$\begin{aligned} \text{Proof: } u_{k+1} &= \frac{2u_k - 1}{3} \\ &= \frac{2 \times 3\left(\frac{2}{3}\right)^k - 2 - 1}{3} \\ &= \frac{2 \times 3\left(\frac{2}{3}\right)^k - 3}{3} \\ &= 2 \times \left(\frac{2}{3}\right)^k - 1 \\ &= \frac{2}{3} \times 3 \times \left(\frac{2}{3}\right)^k - 1 \\ &= 3\left(\frac{2}{3}\right)^{k+1} - 1 \end{aligned}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k+1)$ is true, therefore by induction, $p(n)$ is true.

Exercise 1J

1 $8! - 7! = 8 \times 7! - 7! = 7 \times 7!$

$$10! - 9! = 10 \times 9! - 9! = 9 \times 9!$$

$$5! - 4! = 5 \times 4! - 4! = 4 \times 4!$$

$$95! - 94! = 95 \times 94! - 94! = 94 \times 94!$$

$$(n+1)! - n! = (n+1)n! - n! = n \times n!$$

2 a $\frac{4!}{6!} = \frac{1}{6 \times 5} = \frac{1}{30}$

b $\frac{5! \times 3!}{6!} = \frac{3!}{6} = 1$

c $\frac{8! \times 6!}{5!} = 8! \times 6 = 241920$

3 a $\frac{n! + (n-1)!}{(n+1)!} = \frac{n(n-1)! + (n-1)!}{(n+1)n(n-1)!} = \frac{n+1}{(n+1)n} = \frac{1}{n}$

b $\frac{n! - (n-1)!}{(n-2)!} = \frac{n(n-1)(n-2)! - (n-1)(n-2)!}{(n-2)!}$
 $= n(n-1) - (n-1)$
 $= (n-1)(n-1)$
 $= (n-1)^2$

c $\frac{(n!)^2 - 1}{n! + 1} = \frac{(n! - 1)(n! + 1)}{n! + 1} = n! - 1$

4 $\frac{(2n+2)!(n!)^2}{[(n+1)!]^2(2n)!} = \frac{(2n+2)(2n+1)(2n)!(n!)^2}{(n+1)^2(n!)^2(2n)!}$
 $= \frac{2(n+1)(2n+1)}{(n+1)^2}$
 $= \frac{2(2n+1)}{(n+1)}$

Exercise 1K

- 1 $26 \times 25 \times 24 = 15\,600$
- 2 **a** $12! = 479\,001\,600$
b $4! \times 3! \times 4! \times 2! \times 3! = 41\,472$
- 3 $\binom{8}{4} = 70$ weeks
- 4 **a** $\binom{20}{4} = 4845$
b $4845 - \binom{8}{4} - \binom{12}{4} = 4845 - 70 - 495 = 4280$
- 5 **a** $6 \times 7 \times 7 \times 4 = 1176$
b must end in 0 $6 \times 7 \times 7 \times 1 = 294$
c ending in 0 $6 \times 5 \times 4 \times 1 = 120$
 ending in 2, 4 or 6 $5 \times 5 \times 4 \times 3 = 300$
 $120 + 300 = 420$
- 6 $26^3 \times 10^3 = 17\,576\,000$

Exercise 1L

- 1 **a** $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ $\binom{n}{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!}$
 $\therefore \binom{n}{r} = \binom{n}{n-r}$
b $\binom{n+1}{r} = \frac{(n+1)!}{(n+1-r)!r!}$
 $\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$
 $= \frac{(n-r+1)n! + m!}{(n-r+1)!r!}$
 $= \frac{n \times n! + n!}{(n-r+1)!r!}$
 $= \frac{n!(n+1)}{(n-r+1)!r!}$
 $= \frac{(n+1)!}{(n-r+1)!r!}$
 $\therefore \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$
- 2 **a** $(1+2x)^{11} = 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + \binom{11}{3}(2x)^3 + \dots$
 $= 1 + 22x + 220x^2 + 1320x^3 + \dots$
b $(1-3x)^7 = 1 + \binom{7}{1}(-3x) + \binom{7}{2}(-3x)^2 + \binom{7}{3}(-3x)^3 + \dots$
 $= 1 - 21x + 189x^2 - 945x^3 + \dots$

$$\begin{aligned} \text{c } (2+5x)^5 &= 2^5 + \binom{5}{1}2^4(5x) + \binom{5}{2}2^3(5x)^2 + \binom{5}{3}2^2(5x)^3 + \dots \\ &= 32 + 400x + 2000x^2 + 5000x^3 + \dots \\ \text{d } \left(2 - \frac{x}{3}\right)^9 &= 2^9 + \binom{9}{1}2^8\left(-\frac{x}{3}\right) + \binom{9}{2}2^7\left(-\frac{x}{3}\right)^2 + \binom{9}{3}2^6\left(-\frac{x}{3}\right)^3 + \dots \\ &= 512 - 768x + 512x^2 - \frac{1792}{9}x^3 + \dots \end{aligned}$$

- 3 **a** $(1-4x)^7$ 4th term $= \binom{7}{3}(-4x)^3 = -2240x^3$
b $\left(1 - \frac{x}{2}\right)^{20}$ 3rd term $= \binom{20}{2}\left(-\frac{x}{2}\right)^2 = \frac{95}{2}x^2$
c $(2a-b)^8$ 4th term $= \binom{8}{3}(2a)^5(-b)^3 = -1792a^5b^3$
- 4 $\binom{12}{4}(2x)^8\left(\frac{1}{x^2}\right)^4 = 126720$
- 5 $\left(2 + \frac{x}{5}\right)^5 = 2^5 + \binom{5}{1}2^4 \cdot \frac{x}{5} + \binom{5}{2}2^3 \cdot \frac{x^2}{25} + \binom{5}{3}2^2 \cdot \frac{x^3}{125} + \binom{5}{4}2 \cdot \frac{x^4}{625} + \frac{x^5}{3125}$
 $= 32 + 16x + \frac{80x^2}{25} + \frac{40x^3}{125} + \frac{2x^4}{125} + \frac{x^5}{3125}$
 $= 32 + 16x + \frac{16x^2}{5} + \frac{8x^3}{25} + \frac{2x^4}{125} + \frac{x^5}{3125}$
 $(2.01)^5 = \left(2 + \frac{0.05}{5}\right)^5 = 32 + 0.8 + 0.008 + 0.00004 + 0.0000001 + \dots$
 $= 32.80804$ (5 dp)
- 6 **a** $(\sqrt{2} - \sqrt{3})^4 = 4 - 4 \times 2\sqrt{2} \times \sqrt{3} + 6 \times 2 \times 3 - 4 \times \sqrt{2} \times 3\sqrt{3}$
 $= 4 - 8\sqrt{6} + 36 - 12\sqrt{6} + 9$
 $= 49 - 20\sqrt{6}$
b $(\sqrt{2} + \frac{1}{\sqrt{5}})^3 = 2\sqrt{2} + 3 \times 2 \times \frac{1}{\sqrt{5}} + 3\sqrt{2} \times \frac{1}{\sqrt{5}} + \frac{1}{5\sqrt{5}}$
 $= \frac{13\sqrt{2}}{5} + \frac{31}{5\sqrt{5}} = \frac{13}{5}\sqrt{2} + \frac{31}{5}\sqrt{5}$
c $(1 + \sqrt{7})^5 - (1 - \sqrt{7})^5 = 2 \times 5 \times \sqrt{7} + 2 \times 10 \times (\sqrt{7})^3 + 2 \times (\sqrt{7})^5$
 $= 10\sqrt{7} + 140\sqrt{7} + 98\sqrt{7}$
 $= 248\sqrt{7}$
- 7 **a** $a^2 - b^2 = x^2 + 2xy + y^2 - (x^2 - 2xy + y^2)$
 $= 4xy$
 $= 4 \frac{(a+b)}{2} \cdot \frac{(a-b)}{2}$ (using $2x = a + b$ and $2y = a - b$)
 $= (a+b)(a-b) = (a-b)(a+b)$

$$\begin{aligned}
 \text{b } a^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
 b^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \Rightarrow a^3 - b^3 = 6x^2y + 2y^3 \\
 &= 2y(3x^2 + y^2) = (a - b)(3x^2 + y^2) \\
 &= (a - b) \left[\frac{3(a+b)^2}{2^2} + \left(\frac{a-b}{2} \right)^2 \right] \\
 &= \frac{(a-b)}{4} [3a^2 + 6ab + 3b^2 + a^2 - 2ab + b^2] \\
 &= \frac{(a-b)}{4} (4a^2 + 4ab + 4b^2) \\
 &= (a - b)(a^2 + ab + b^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } a^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
 b^4 &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\
 \Rightarrow a^4 - b^4 &= 8x^3y + 8xy^3 = 8xy(x^2 + y^2) \\
 &= 8 \frac{(a+b)}{2} \frac{(a-b)}{2} (x^2 + y^2) \\
 &= 2(a-b)(a-b) \left[\left(\frac{a+b}{2} \right)^2 + \left(\frac{a-b}{2} \right)^2 \right] \\
 &= 2(a-b)(a+b) \left[\frac{a^2}{2} + \frac{b^2}{2} \right] \\
 &= (a-b)(a+b)(a^2 + b^2)
 \end{aligned}$$

$$\text{d } (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$$

$$\text{e } \text{Let } p(n) \text{ be } a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$$

When $n = 1$, $a^1 - b^1 = a - b$ so $p(1)$ is true.

Assume $p(n)$ is true for $n = k$ i.e. $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \dots + b^{k-1})$

Prove $p(n)$ is true for $n = k + 1$:

$$\begin{aligned}
 a^{k+1} - b^{k+1} &= a \times a^k - b^{k+1} \\
 &= a[(a-b)(a^{k-1} + a^{k-2}b + \dots + b^{k-1})] + a \times b^k - b^{k+1} \\
 &= (a-b)[a^k + a^{k-1}b + \dots + ab^{k-1}] + ab^k - b^{k+1} \\
 &= (a-b)(a^k + a^{k-1}b + \dots + ab^{k-1} + (a-b)b^k + b^{k+1} - b^{k+1}) \\
 &= (a-b)(a^k + a^{k-1}b + \dots + ab^{k-1} + b^k)
 \end{aligned}$$

$\therefore p(k+1)$ is true.

So, since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, therefore by induction $p(n)$ is true.



Review exercise

$$\begin{aligned}
 \text{1 } u_2 &= 16 & S_3 &= 84 \\
 u_1 r &= 16 & u_1 + u_1 r + u_1 r^2 &= 84 \\
 u_1 &= \frac{16}{r} & u_1(1 + r + r^2) &= 84 \\
 & & \frac{16}{r}(1 + r + r^2) &= 84 \\
 & & 16 + 16r + 16r^2 &= 84r \\
 & & 16r^2 - 68r + 16 &= 0 \\
 & & 4r^2 - 17r + 4 &= 0 \\
 & & (4r-1)(r-4) &= 0 \quad r = \frac{1}{4} \text{ or } 4 \\
 \text{if } r &= \frac{1}{4}, u_1 &= 64 & \quad 64, 16, 4 \\
 \text{if } r &= 4, u_1 &= 4 & \quad 4, 16, 64
 \end{aligned}$$

$$\begin{aligned}
 \text{2 } 1 + 3 + 4 + 6 + 7 + 9 + 10 + 12 + \dots + 46 \\
 &= (1 + 4 + 7 + \dots + 46) + (3 + 6 + 9 + \dots + 45) \\
 &= \frac{16}{2}(1 + 46) + \frac{15}{2}(3 + 45) \\
 &= 376 + 360 = 736
 \end{aligned}$$

$$\text{3 } c - b = b - a \quad \frac{b}{a} = \frac{a}{c} \quad a + b + c = \frac{-9}{2} \quad (3)$$

$$\therefore a + c = 2b \quad (1) \quad \therefore bc = a^2 \quad (2)$$

$$\text{substitute (1) in (3)} \quad 2b + b = \frac{-9}{2}$$

$$3b = \frac{-9}{2} \quad \therefore b = \frac{-3}{2}$$

$$a + c = -3 \quad \frac{-3}{2}c = a^2$$

$$c = -3 - a \quad \therefore \frac{-3}{2}(-3 - a) = a^2$$

$$9 + 3a = 2a^2$$

$$2a^2 - 3a - 9 = 0$$

$$(2a + 3)(a - 3) = 0$$

$$a = \frac{-3}{2} \text{ or } 3$$

$$a \neq \frac{-3}{2} \text{ since } a \neq b \quad \therefore a = 3, \quad c = -6$$

The three numbers are $3, \frac{-3}{2}, -6$

$$\text{4 } 1, 3, 7, 15, 31, 63$$

$$p(n): u_n = 2^n - 1$$

$$\text{Step 1: when } n = 1, u_1 = 1 = 2^1 - 1$$

$\therefore p(1)$ is true.

$$\text{Step 2: assume } p(k) \text{ i.e. } u_k = 2^k - 1$$

$$\text{Step 3: prove } p(k+1) \text{ i.e. } u_{k+1} = 2^{k+1} - 1$$

$$\begin{aligned}
 \text{proof: } u_{k+1} &= 2u_k + 1 \\
 &= 2(2^k - 1) + 1 \\
 &= 2^{k+1} - 2 + 1 \\
 &= 2^{k+1} - 1
 \end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

$$\text{5 } p(n): 3^{2n} - 8n - 1 = 64A \quad (A \in \mathbb{Z}, \in \mathbb{Z}^+)$$

$$\text{Step 1: when } n = 1, 3^2 - 8 - 1 = 0$$

$\therefore p(1)$ is true.

$$\text{Step 2: assume } p(k) \text{ i.e. } 3^{2k} - 8k - 1 = 64A$$

$$\text{Step 3: prove } p(k+1) \text{ i.e. } 3^{2(k+1)} - 8(k+1) - 1 = 64B \quad (B \in \mathbb{Z})$$

$$\begin{aligned}
 \text{Proof: } 3^{2(k+1)} - 8(k+1) - 1 &= 3^{2k}(3^2) - 8k - 9 \\
 &= 9(64A + 8k + 1) - 8k - 9 \\
 &= 576A + 72k + 9 - 8k - 9 \\
 &= 576A + 64k \\
 &= 64(9A + k) \\
 &= 64B
 \end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

$$6 \quad a \quad \binom{n+1}{4} = \frac{(n+1)!}{(n-3)!4!} \quad b \quad \binom{n-1}{2} = \frac{(n-1)!}{(n-3)!2!}$$

$$c \quad \frac{(n+1)!}{(n-3)!4!} = \frac{6(n-1)!}{(n-3)!2!}$$

$$\frac{(n+1)n}{24} = 3$$

$$n^2 + n = 72$$

$$n^2 + n - 72 = 0$$

$$(n+9)(n-8) = 0$$

$\therefore n = 8$ (n cannot be negative)

$$7 \quad (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$$

$$a \quad \text{Let } x = 1, \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} + \dots + \binom{n}{n} = 2^n$$

$$b \quad \text{Let } x = -1, \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^r \binom{n}{r} + \dots + (-1)^n \binom{n}{n} = 0$$

$$2 \quad a \quad \frac{14!}{3!2!2!2!} = 908\,107\,200$$

b Consider 5 digit and 6 digit numbers ending in 0 or 5.

5 digit numbers:

$$4 \times 6 \times 6 \times 6 \times 2 = 1728$$

6 digit numbers:

$$5 \times 6 \times 6 \times 6 \times 6 \times 2 = 12\,960$$

$$1728 + 12\,960 = 14\,688$$

$$c \quad 4! \times (2!)^4 = 384$$

$$3 \quad \begin{array}{|c|c|} \hline M & W \\ \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} \quad \binom{6}{2} \times \binom{4}{3} + \binom{6}{1} \times \binom{4}{4} = 15 \times 4 + 6 \times 1 = 66$$

$$4 \quad \binom{8}{6} = (x^3)^2 \left(-\frac{3}{x}\right)^6 = 20412$$



Review exercise

$$1 \quad a \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = a^2 \quad a^2 = \frac{1}{2} \quad a = \frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = b^2 \quad b^2 = \frac{1}{4} \quad b = \frac{1}{2}$$

$$\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = c^2 \quad c^2 = \frac{1}{8} \quad c = \frac{1}{2\sqrt{2}}$$

b The spiral consists of 1.5 of the sides of the first eight squares and one of the sides of the ninth square.

$$\begin{aligned} \text{length} &= 1.5 \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots + \left(\frac{1}{\sqrt{2}}\right)^7 \right) + \left(\frac{1}{\sqrt{2}}\right)^8 \\ &= 1.5 \left(\frac{1 - \left(\frac{1}{\sqrt{2}}\right)^8}{1 - \frac{1}{\sqrt{2}}} \right) + \frac{1}{16} = 4.86 \end{aligned}$$

$$c \quad \text{length} = 1.5 \left(\frac{1}{1 - \frac{1}{\sqrt{2}}} \right) = 5.12$$

d The spiral consists of 8 triangles

$$\text{Area} = \frac{1}{2} \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{4}\right)^2 + \dots \right) \text{ to 8 terms}$$

$$= \frac{1}{2} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right) \text{ to 8 terms}$$

$$\text{Area} = \frac{1}{2} \left(\frac{\frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^8 \right)}{1 - \frac{1}{2}} \right) = 0.249$$

$$e \quad \text{Area} = \frac{1}{2} \left(\frac{\frac{1}{4}}{1 - \frac{1}{2}} \right) = 0.25$$

$$5 \quad \text{Coefficients are } \binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$$

$$\frac{n!}{(n-r-1)!(r-1)!} - \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!r!} - \frac{n!}{(n-r+1)!(r-1)!}$$

Divide by $n!$ and multiply by $(r+1)!(n-r+1)!$

$$(n-r+1)(n-r) - (r+1)$$

$$(n-r+1) = (r+1)$$

$$(n-r+1) - (r+1)r$$

$$(n-r+1)(n-r) - 2$$

$$(r+1)(n-r+1) +$$

$$(r+1)r = 0$$

$$n^2 - rn - rn + r^2 + n - r - 2rn + 2r^2 - 2r - 2n + 2r - 2 + r^2 + r = 0$$

$$n^2 - 4rn + 4r^2 - n - 2 = 0$$

$$n^2 + 4r^2 - 2 - n(4r+1) = 0$$

$$n = 14, \quad 196 + 4r^2 - 2 - 14(4r+1) = 0$$

$$4r^2 - 56r + 180 = 0$$

$$r^2 - 14r + 45 = 0$$

$$(r-5)(r-9) = 0$$

$$r = 5 \text{ or } 9$$

$$\text{The coefficients are } \binom{14}{4}, \binom{14}{5}, \binom{14}{6} \text{ or } \binom{14}{8}, \binom{14}{9}, \binom{14}{10}$$

Both sets give 1001, 2002, 3003.

2

Mathematics as a language

Answers

Skills check

1 $y = x^2 - 3x - 1$

$$y = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

Vertex is $\left(\frac{3}{2}, -\frac{13}{4}\right)$ Axis of symmetry is $x = \frac{3}{2}$

2 a $3x + 4 = 0, x = -\frac{4}{3}$

b $3x^2 - 2x - 1 = 0$
 $(3x + 1)(x - 1) = 0$
 $x = -\frac{1}{3}$ or 1

3 a $y - 3 = \sqrt{x - 2}$
 $\therefore x - 2 = (y - 3)^2$
 $\therefore x = (y - 3)^2 + 2$

b $y = \frac{2x - 1}{3x + 2}$
 $3xy + 2y = 2x - 1$
 $2x - 3xy = 2y + 1$
 $x(2 - 3y) = 2y + 1$
 $x = \frac{2y + 1}{2 - 3y}$

Exercise 2A

- 1 a function, domain = $\{0, 1, 2, 3\}$,
range = $\{-1, 1, 2, 3\}$
b function, domain = $\{-3, -2, -1, 0\}$,
range = $\{0\}$
c not a function
d not a function
- 2 a function, domain = $\{x | -3 \leq x \leq 3\}$,
range = $\{y | 0 \leq y \leq 3\}$
b not a function
c not a function
d function, domain = $\{x | x \geq -1\}$,
range = $\{y | y \geq 0\}$

Exercise 2B

- 1 $y^2 = x \Rightarrow y = \pm \sqrt{x}$
one value of x gives 2 values of y
eg. if $x = 4, y \pm 2 \therefore$ not a function.
 $y = \sqrt{x}, \sqrt{x}$ is the positive square root of x
 $\therefore$ each value of x gives just one value of y

- 2 a $y = x^2 - 4x + 2$ domain is $x \in \mathbb{R}$
 $y = (x - 2)^2 - 2$ range = $\{y | y \geq -2\}$
b $y = -(x + 2)^2 - 3$ domain is $x \in \mathbb{R}$
range = $\{y | y \leq -3\}$
c $y = \sqrt{x + 2}$ $x + 2 \geq 0$ domain = $\{x | x \geq -2\}$
 $x \geq -2$ range = $\{y | y \geq 0\}$
d $y = \sqrt{3 - x}$ $3 - x \geq 0$ domain = $\{x | x \leq 3\}$
 $x \leq 3$ range = $\{y | y \geq 0\}$
e $y = -3x^2 + 6x - 1$ domain is $x \in \mathbb{R}$
 $= -3(x^2 - 2x) - 1$
 $= -3[(x - 1)^2 - 1] - 1$
 $= -3(x - 1)^2 + 2$ range = $\{y | y \leq 2\}$
f $y = \sqrt{4 - 2x}$ $4 - 2x \geq 0$ domain = $\{x | x \leq 2\}$
 $x \leq 2$ range = $\{y | y \geq 0\}$

Exercise 2C

- 1 $y = -|x|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$
2 $y = |2x + 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq 0\}$
3 $y = -|2x + 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$
4 $y = 2|x - 1|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq 0\}$
5 $y = -\frac{1}{2}|3x + 2|$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 0\}$
6 $y = |x + 4| - 2$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq -2\}$
7 $y = -2|x - 1| + 1$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \leq 1\}$
8 $y = 3|1 - 2x| - 2$ domain = $\{x | x \in \mathbb{R}\}$
range = $\{y | y \geq -2\}$

Exercise 2D

- 1 $y = \frac{1}{3x + 2}$ domain = $\left\{x \mid x \neq -\frac{2}{3}\right\}$,
range = $\{y | y \neq 0\}$
2 $y = -\frac{1}{2 - x}$ domain = $\{x | x \neq 2\}$,
range = $\{y | y \neq 0\}$
3 $y = \frac{3}{3 - x}$ domain = $\{x | x \neq 3\}$,
range = $\{y | y \neq 0\}$

$$\begin{aligned}
 4 \quad y &= -\frac{5}{6x+3} & \text{domain} &= \left\{x \mid x \neq -\frac{1}{2}\right\}, \\
 & & \text{range} &= \{y \mid y \neq 0\} \\
 5 \quad y &= \frac{1+2x}{1-2x} & \text{domain} &= \left\{x \mid x \neq \frac{1}{2}\right\}, \\
 & & \text{range} &= \{y \mid y \neq -1\} \\
 6 \quad y &= -\frac{2-3x}{1+x} & \text{domain} &= \{x \mid x \neq -1\}, \\
 & & \text{range} &= \{y \mid y \neq 3\}
 \end{aligned}$$

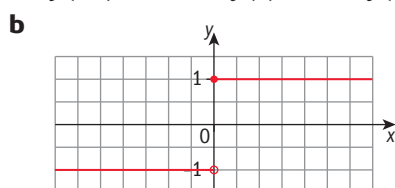
Exercise 2E

$$\begin{aligned}
 1 \quad y &= \frac{1}{|x+1|} & \text{domain} &= \{x \mid x \neq -1\}, \text{ range} = \{y \mid y > 0\} \\
 2 \quad y &= \frac{-2}{|x-1|} & \text{domain} &= \{x \mid x \neq 1\}, \text{ range} = \{y \mid y < 0\} \\
 3 \quad y &= \frac{x}{|x|} & \text{domain} &= \{x \mid x \neq 0\} \\
 & & \text{If } x > 0, y &= \frac{x}{x} = 1 \\
 & & \text{If } x < 0, y &= \frac{x}{-x} = -1 \\
 & & \therefore \text{range} &= \{-1, 1\} \\
 4 \quad y &= \frac{-2}{\sqrt{1-x}} & 1-x > 0 & \therefore x < 1 \\
 & & \text{domain} &= \{x \mid x < 1\}, \text{ range} = \{y \mid y < 0\}
 \end{aligned}$$

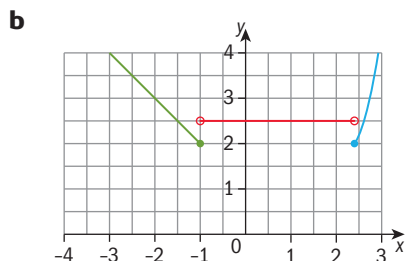
$$\begin{aligned}
 5 \quad a \quad & \text{For } f \text{ to be real, } \sqrt{\frac{1}{x^2}-2} > 0 \\
 & \Rightarrow \frac{1}{x^2} - 2 > 0 \\
 & \Rightarrow \frac{1}{x^2} - 2 > 0 \\
 & \therefore x^2 < \frac{1}{2} \\
 & \text{so domain} = \left\{x \mid -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, x \neq 0\right\} \\
 b \quad & \text{Range} = \{y \mid y > 0\}
 \end{aligned}$$

Exercise 2F

$$\begin{aligned}
 1 \quad y &= \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \\
 a \quad f(-3) &= -1 \quad f(0) = 1 \quad f(\pi) = 1 \quad f(4) = 1
 \end{aligned}$$



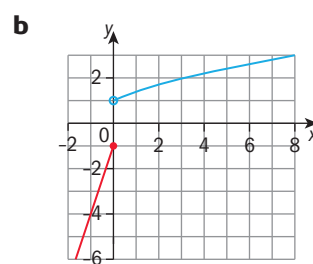
$$\begin{aligned}
 c \quad \text{domain} &= \{x \mid x \in \mathbb{R}\} \quad \text{range} = \{-1, 1\} \\
 2 \quad y &= \begin{cases} 1-x, & x \leq -1 \\ 2.5, & -1 < x < \sqrt{6} \\ x^2-4, & x \geq \sqrt{6} \end{cases} \\
 a \quad f(-3) &= 4 \quad f(0) = 2.5 \quad f(\sqrt{6}) = 2 \quad f(3) = 5
 \end{aligned}$$



c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{y \mid y \geq 2\}$

$$3 \quad f(x) = \begin{cases} 3x-1, & x \leq 0 \\ \sqrt{x+1}, & x > 0 \end{cases}$$

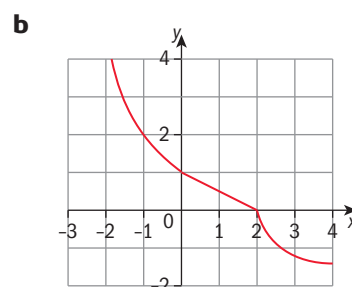
a $f(-1) = -4 \quad f(0) = -1 \quad f(1) = \sqrt{2} \quad f(8) = 3$



c domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \leq -1 \text{ or } y > 1\}$

$$4 \quad g(x) = \begin{cases} x^2+1, & x \leq 0 \\ -\frac{1}{2}x+1, & 0 < x \leq 2 \\ -\sqrt{x-2}, & x > 2 \end{cases}$$

a $f(-2) = 5 \quad f(1) = \frac{1}{2} \quad f(2) = 0 \quad f(3) = -1$



c domain = $\{x \mid x \in \mathbb{R}\}$ range = $\{y \mid y \in \mathbb{R}\}$

Exercise 2G

$$\begin{aligned}
 1 \quad f(x) &= 4 - x^2 & a \quad & \text{many-to-one} \\
 b \quad f(-x) &= 4 - (-x)^2 = 4 - x^2 = f(x) & \therefore & \text{even}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad g(x) &= x^3 + 3x & a \quad & \text{one-to-one} \\
 b \quad g(-x) &= (-x)^3 + 3(-x) = -x^3 - 3x = -g(x) & \therefore & \text{odd}
 \end{aligned}$$

$$3 \quad h(x) = \frac{-3}{2x} \quad a \quad \text{one-to-one}$$

b $h(-x) = \frac{-3}{2(-x)} = \frac{3}{2x} = -h(x) \quad \therefore \text{odd}$

$$\begin{aligned}
 4 \quad p(x) &= x^3 + 4x + 1 & a \quad & \text{one-to-one} \\
 b \quad p(-x) &= (-x)^3 + 4(-x) + 1 = -x^3 - 4x + 1 & \neq p(x) \text{ or } -p(x) & \therefore \text{neither odd nor even}
 \end{aligned}$$

$$5 \quad r(x) = \begin{cases} -1 & 0 \leq x < \pi \\ 1 & \pi \leq x < 2\pi \\ -1 & 2\pi < x < 3\pi \end{cases} \quad a \quad \text{many-to-one}$$

b If $0 \leq x < 3\pi$, $r(-x)$ is not defined
 $\therefore$ neither even nor odd

$$\begin{aligned}
 6 \quad q(x) &= 2x^3 - 4x & a \quad & \text{many-to-one} \\
 b \quad q(-x) &= 2(-x)^3 - 4(-x) = -2x^3 + 4x = -q(x) & \therefore & \text{odd}
 \end{aligned}$$

- 7 $w(x) = x - 2x^3 + x^5$ **a** many-to-one
b $w(-x) = -x - 2(-x)^3 + (-x)^5 = -x + 2x^3 - x^5$
 $= -w(x)$
 $\therefore$ odd
- 8 $t(x) = 4x^4 - x$ **a** many-to-one
b $t(-x) = 4(-x)^4 - (-x) = 4x^4 + x \neq t(x)$ or $-t(x)$
 $\therefore$ neither even nor odd
- 9 $f(x) = 0$ is both even and odd

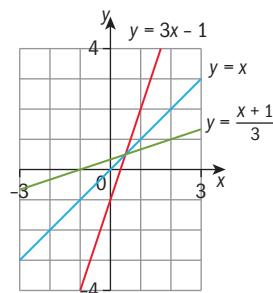
Exercise 2H

- 1 $f(x) = 2x$ $g(x) = \sqrt{x}$
 domain of f is all real numbers
 domain of g is all non-negative real numbers
a $2g(x) - f(x)$ domain is all non-negative real numbers
b $f(x) \cdot g(x)$ domain is all non-negative real numbers
c $\left(\frac{g}{f}\right)(x)$ domain is all positive real numbers
- 2 $f(x) = |x + 1|$ $g(x) = \sqrt{x^2 - 4}$
 domain of f is all real numbers
 domain of $g = \{x \mid x \leq -2, x \geq 2\}$
 domain of $\left(\frac{f}{g}\right)(x) = \{x \mid x < -2, x > 2\}$
- 3 $f(x) = x^2 + 2x - 1$ $g(x) = 1 - 2x - 3x^2$
a $f(g(0)) = f(1) = 2$
b $g(f(-1)) = g(-2) = -7$
c $f(f(0)) = f(-1) = -2$
d $g(g(x)) = g(1 - 2x - 3x^2)$
 $= 1 - 2(1 - 2x - 3x^2) - 3(1 - 2x - 3x^2)^2$
 $= 1 - 2 + 4x + 6x^2 - 3(1 - 4x + 6x^2 - 2x + 4x^2 + 6x^3 - 3x^2 + 6x^3 + 9x^4)$
 $= -1 + 4x + 6x^2 - 3 + 12x + 6x^2 - 36x^3 - 27x^4$
 $= -4 + 16x + 12x^2 - 36x^3 - 27x^4$
- 4 $f(x) = 1 - 2x$ $g(x) = x^2 - 1$ $h(x) = \sqrt{2x + 4}$
a, b **i** $f(g(x)) = f(x^2 - 1)$
 $= 1 - 2(x^2 - 1) = 3 - 2x^2$
 domain $= \{x \mid x \in \mathbb{R}\}$ range $= \{y \mid y \leq 3\}$
ii $g(h(x)) = g(\sqrt{2x + 4}) = 2x + 4 - 1$
 $= 2x + 3$ domain
 $=$ domain of $h = \{x \mid x \geq -2\}$ range
 $= \{y \mid y \geq -1\}$
iii $f(h(x)) = f(\sqrt{2x + 4}) = 1 - 2\sqrt{2x + 4}$
 domain $= \{x \mid x \geq -2\}$ range $= \{y \mid y \leq 1\}$
iv $h(g(x)) = h(x^2 - 1) = \sqrt{2(x^2 - 1) + 4}$
 $= \sqrt{2x^2 + 2}$
 domain $= \{x \mid x \in \mathbb{R}\}$ range $= \{y \mid y \geq \sqrt{2}\}$
- d** $f(g(1)) = 3 - 2(1)^2 = 1$
 $h(f(g(1))) = h(1) = \sqrt{6}$

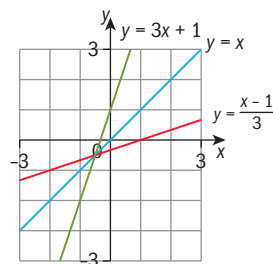
- 5 e.g. $f(x) = x - 2$ $g(x) = x^2$
 6 e.g. $g(x) = 2x - 3$ $h(x) = \sqrt{x}$

Exercise 2I

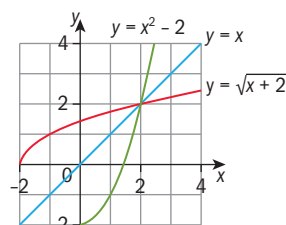
- 1 $y = 3x - 1$ $x = 3y - 1$
 $\therefore y = \frac{x+1}{3}$ $f^{-1}(x) = \frac{x+1}{3}$
 $f(f^{-1}(x)) = 3\left(\frac{x+1}{3}\right) - 1 = x + 1 - 1 = x$
 $f^{-1}(f(x)) = \frac{(3x-1)+1}{3} = \frac{3x}{3} = x$



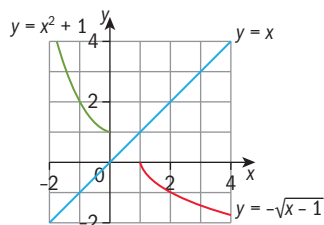
- 2 $y = \frac{x-1}{3}$
 $x = \frac{y-1}{3} \therefore y = 3x + 1$ $f^{-1}(x) = 3x + 1$
 $f(f^{-1}(x)) = \frac{(3x+1)-1}{3} = \frac{3x}{3} = x$



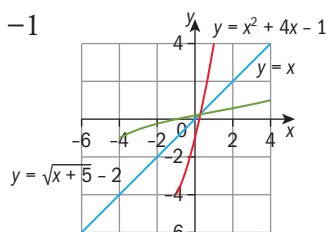
- 3 $y = x^2 - 2, x \geq 0$
 $x = y^2 - 2 \therefore y = \sqrt{x+2}, x \geq -2$
 $f^{-1}(x) = \sqrt{x+2}, x \geq -2$
 $f(f^{-1}(x)) = (\sqrt{x+2})^2 - 2 = x + 2 - 2 = x$
 $f^{-1}(f(x)) = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = x$



- 4 $y = x^2 + 1$ $x \leq 0$
 $x = y^2 + 1$
 $\therefore y = -\sqrt{x-1}, x \geq 1$ $f^{-1}(x) = -\sqrt{x-1}, x \geq 1$
 $f(f^{-1}(x)) = (-\sqrt{x-1})^2 + 1 = x - 1 + 1 = x$
 $f^{-1}(f(x)) = -\sqrt{x^2 + 1 - 1} = -\sqrt{x^2} = x$



5 $y = x^2 + 4x - 1 \quad x \geq -1$
 $x = y^2 + 4y - 1$
 $y^2 + 4y = x + 1$
 $y^2 + 4y + 4 = x + 5$
 $(y + 2)^2 = x + 5$
 $y + 2 = \sqrt{x + 5}$
 $y = \sqrt{x + 5} - 2$
 $f^{-1}(x) = \sqrt{x + 5} - 2, \quad x \geq -4$ (since range of $f(x)$ is $y \geq -4$)

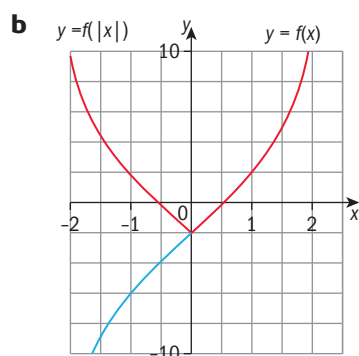
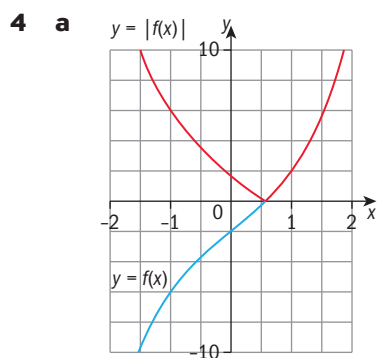
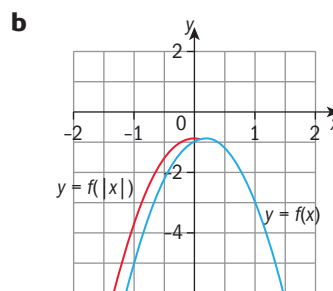
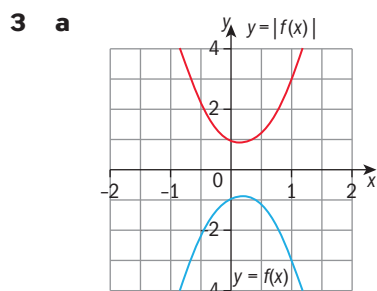
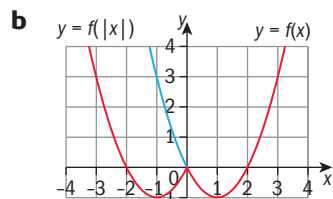
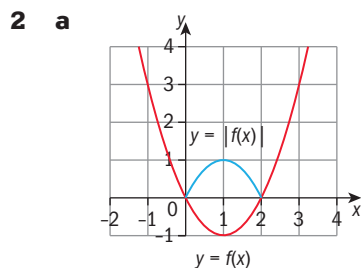
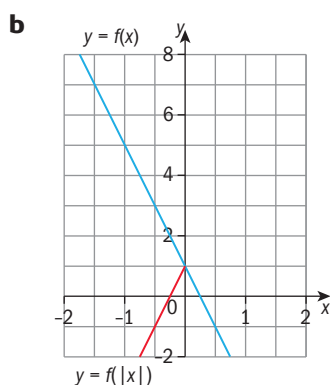
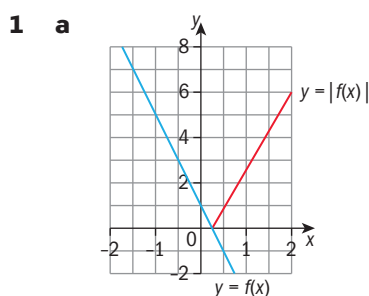


6 $y = 1 - 2x \quad x = 1 - 2y$
 $2y = 1 - x$
 $y = \frac{1-x}{2}$
 $\therefore y = 1 - 2x$ is not its own inverse

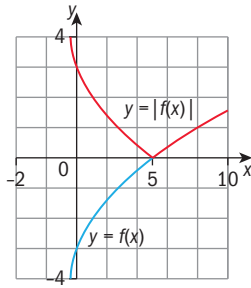
7 $f(x) = 3x \quad g(x) = 2x + 1$
 $f^{-1}(x) = \frac{x}{3} \quad g^{-1}(x) = \frac{x-1}{2}$
 $g \circ f(x) = g(3x) = 2(3x) + 1 = 6x + 1$
 $(g \circ f)^{-1}(x) = \frac{x-1}{6} \quad f^{-1} \circ g^{-1}(x) = f^{-1}\left(\frac{x-1}{2}\right) = \frac{x-1}{6}$
 $\therefore f^{-1} \circ g^{-1}(x) = (g \circ f)^{-1}(x)$

8 $y = \frac{2x+1}{x-1}$
 $x = \frac{2y+1}{y-1}$
 $y(x-1) = 2y+1$
 $y(x-2) = 1+x$
 $y = \frac{1+x}{x-2}$
 $\therefore f^{-1}(x) = \frac{1+x}{x-2}, \text{ domain} = \{x \mid x \neq 2\}$

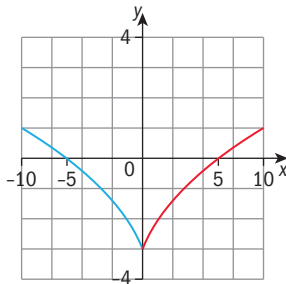
Exercise 2J



5 a

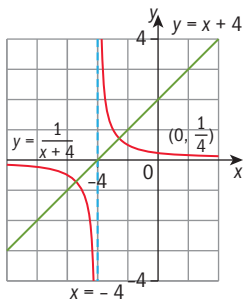


b

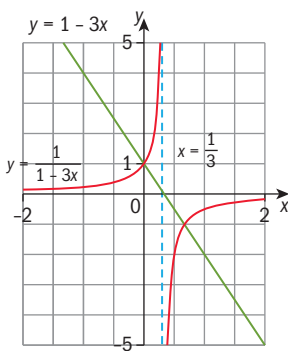


Exercise 2K

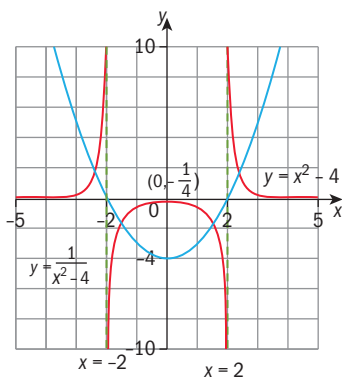
1



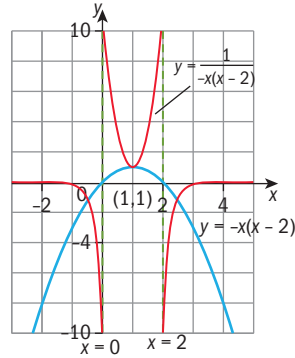
2



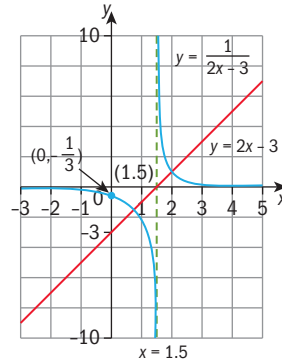
3



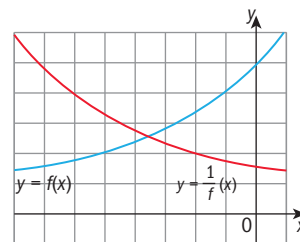
4



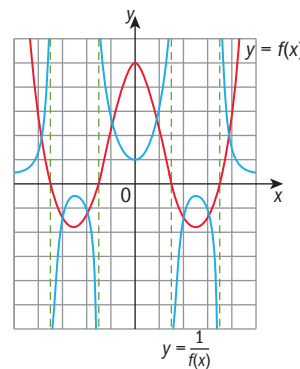
5



6 a



b



Exercise 2L

1 a Translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, i.e. $g(x) = f(x - 3) + 2$

b Translation $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, i.e. $g(x) = f(x + 2) - 1$

c Reflection in the x -axis and translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, i.e. $g(x) = -f(x) - 1$

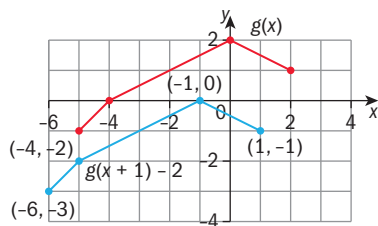
d Horizontal compression by a factor of $\frac{1}{2}$, i.e. $g(x) = f(2x)$

e Reflection in the x -axis and vertical stretch of 2, i.e. $g(x) = -2f(x)$

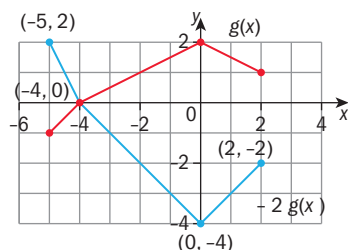
f Reflection in the y -axis and horizontal stretch of 2, i.e. $g(x) = f\left(-\frac{1}{2}x\right)$

2 $g(x) = h(-(x-3))$ or $g(x) = h(-x+3)$

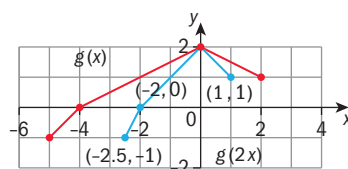
3 a



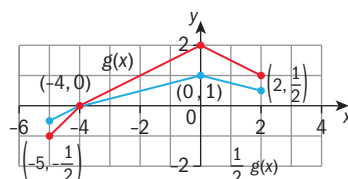
b



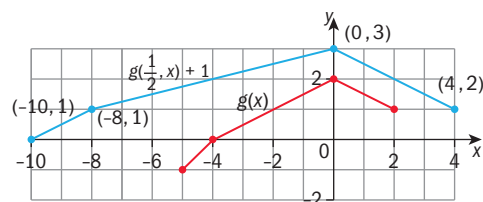
c



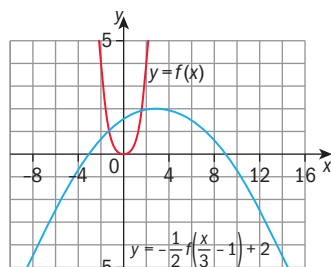
d



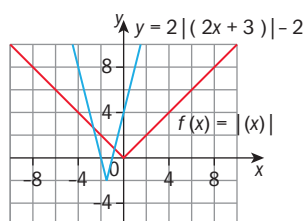
e



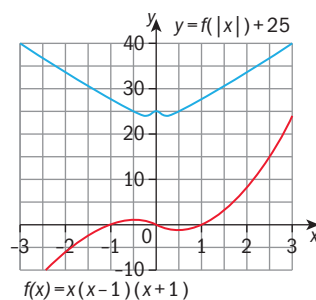
4 a



b



c



5 $y = \frac{2}{3(x+2)} + 3$ $y = \frac{2+9(x+2)}{3(x+2)}$ $y = \frac{9x+20}{3(x+2)}$

Domain = $\{x \mid x \neq -2\}$, Range = $\{y \mid y \neq 3\}$

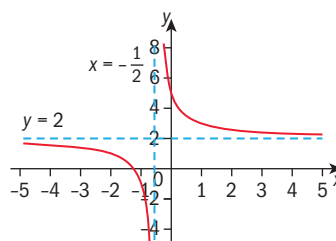
6 a $x = -\frac{1}{2}, y = 2$

b When $x = 0, y = \frac{5}{1} = 5$
 $\Rightarrow$ intercept at $(0, 5)$

When $y = 0, 4x + 5 = 0$

$4x = -5$
 $x = -\frac{5}{4}$
 $\Rightarrow$ intercept at $(-\frac{5}{4}, 0)$

c



d $y = \frac{2(2x+1)+3}{2x+1} = 2 + \frac{3}{2x+1}$
 $\therefore$ if $g(x) = \frac{1}{x}$, then $f(x) = 3g(2x+1) + 2$
 i.e. vertical stretch of 3, then horizontal stretch of $\frac{1}{2}$, then translation $(-\frac{1}{2}, 2)$.



Review exercise

- 1 a function domain = $\{x \mid x \in \mathbb{R}\}$
 range = $\{y \mid y \leq 2\}$
 b not a function
 c not a function
 d function domain = $\{x \mid x \in \mathbb{R}, x \neq \pm 1\}$
 range = $\{y \mid y \leq -0.25 \text{ or } y > 0\}$

2 $f(g(h(3))) = f(g(4)) = f(2) = 3$
 $h^{-1}(g^{-1}(f^{-1}(3))) = h^{-1}(g^{-1}(2)) = h^{-1}(4) = 3$

3 $f \circ f(x) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}-1} = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}$

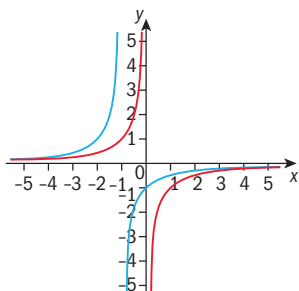
$y = \frac{x-1}{2-x}$ $x = \frac{y-1}{2-y}$ $2x - yx = y - 1$

$y + yx = 2x + 1$ $y(1+x) = 2x + 1$ $y = \frac{2x+1}{x+1}$

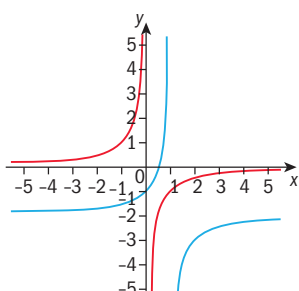
$\therefore (f \circ f)^{-1}(x) = \frac{2x+1}{x+1}$

- 4 $(-5, -2) \rightarrow (-6, 1) \quad (-4, 0) \rightarrow (-5, -3)$
 $(-3, 2) \rightarrow (-4, -7) \quad (-1, -1) \rightarrow (-2, -1)$
 $(3, -3) \rightarrow (2, 3) \quad (8, 2) \rightarrow (7, -7)$

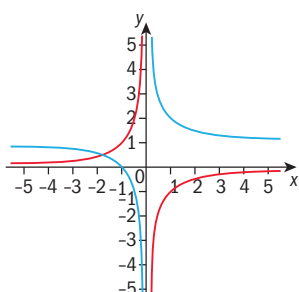
5 a



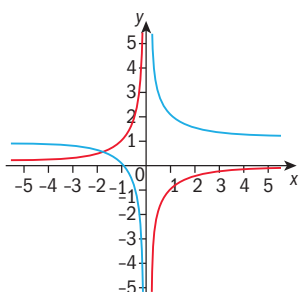
b



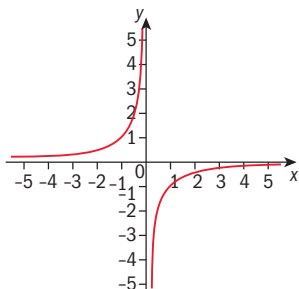
c



d



e



- 6 a Reflection in x -axis $\Rightarrow g(x) = -f(x)$
b Reflection in y -axis $\Rightarrow g(x) = f(-x)$
c Horizontal shift of 3 units to the left, vertical shift of 1 unit down $\Rightarrow g(x) = f(x+3) - 1$
d Reflection in the horizontal line $y = 1$
 $\Rightarrow g(x) = -f(x) + 1$
e $g(x) = \frac{1}{f(x)}$

f Compression by factor of $\frac{1}{2}$ and reflection in x -axis $\Rightarrow g(x) = -f(2x)$

g Reflection in y -axis and vertical shift of 2 units upwards $\Rightarrow g(x) = f(-x) + 2$

7 $f(x)$ is odd, so $f(-x) = -f(x)$

Let $x = 0$, then $f(0) = -f(0)$

i.e. $2f(0) = 0$

$\therefore f(0) = 0$

$\therefore f(x)$ passes through $(0, 0)$



Review exercise

1 a $f(g(x)) = f(x^2) = 3x^2 - 1$

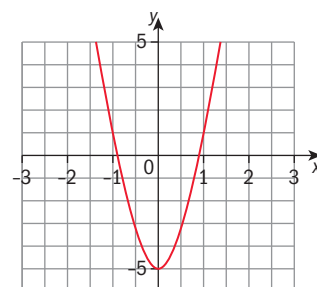
b $h(g(x)) = h(x^2) = \frac{1}{x^2 + 2}$

c $g^{-1}(x) = \sqrt{x} \quad g^{-1}(f(x))$
 $= g^{-1}(3x - 1) = \sqrt{3x - 1}$

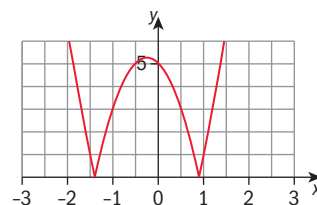
d Find h^{-1} : $x = \frac{1}{y+2} \therefore y+2 = \frac{1}{x} \therefore y = \frac{1}{x} - 2$
 $f(h^{-1}(x)) = f\left(\frac{1}{x} - 2\right) = 3\left(\frac{1}{x} - 2\right) - 1 = \frac{3}{x} - 7$
 $= \frac{3-7x}{x}$

2 $f(x) = \frac{x-1}{x+3} \quad g(x) = x^2$

3 $f(x) = |x| \quad g(x) = 4x^2 + 2x - 5$
 $g(f(x)) = 4|x|^2 + 2|x| - 5$



$f(g(x)) = |4x^2 + 2x - 5|$



4 Vertical stretch of $\frac{9}{5}$ followed by translation $\begin{pmatrix} 0 \\ 32 \end{pmatrix}$

$x = \frac{5}{9}x - \frac{160}{9}$

$9x = 5x - 160$

$4x = -160$ and $\therefore x = -40$

5 $g(x) = \frac{3}{2(x+1)} - 2$
 $= \frac{3-4(x+1)}{2(x+1)} = \frac{-1-4x}{2(x+1)}$

3

The long journey of mathematics

Answers

Skills check

1 a $x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3 \text{ or } 1$

b $x^2 - 11x + 10 = 0$

$(x - 10)(x - 1) = 0$

$x = 1 \text{ or } 10$

c $2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

$x = -\frac{3}{2} \text{ or } 1$

2 a $f(x) + g(x) = 2x^3 - 3$

$$\begin{aligned} \text{b } 2h(x) - 4g(x) + 5f(x) &= 6x^4 - 4x^2 - 10 - 8x^3 \\ &\quad + 4x^2 - 12x + 16 + 5x^2 \\ &\quad - 15x + 5 \\ &= 6x^4 - 8x^3 + 5x^2 - 27x + 11 \end{aligned}$$

$$\begin{aligned} \text{c } \frac{1}{2}h(x) - \frac{2}{5}g(x) &= \frac{3}{2}x^4 - x^2 - \frac{5}{2} - \frac{4}{5}x^3 + \frac{2}{5}x^2 - \frac{6}{5}x + \frac{8}{5} \\ &= \frac{3}{2}x^4 - \frac{4}{5}x^3 - \frac{3}{5}x^2 - \frac{6}{5}x - \frac{9}{10} \end{aligned}$$

Exercise 3A

1 a $2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0 \text{ or } \frac{3}{2}$

b $3x^2 - 75 = 0$

$x^2 = 25$

$x = \pm 5$

c $5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0 \text{ or } \frac{4}{5}$

d $7 + 28x^2 = 0$

no real roots

e $242x^2 + 2x = 0$

$2x(121x + 1) = 0$

$x = 0 \text{ or } -\frac{1}{121}$

f $\sqrt{2}x^2 - \sqrt{8} = 0$

$x^2 = 2$

$x = \pm \sqrt{2}$

g $\pi x^2 - 11x = 0$

$x(\pi x - 11) = 0$

$x = 0 \text{ or } \frac{11}{\pi}$

h $ex^2 - \sqrt{3} = 0$

$x^2 = \frac{\sqrt{3}}{e}$

$x = \pm \sqrt{\frac{\sqrt{3}}{e}}$

2 a $2x^2 + 5x + 2 = 0$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(2)}}{4}$$

$$x = \frac{-5 \pm \sqrt{9}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$x = -2 \text{ or } -\frac{1}{2}$

b $3x^2 - 10x + 3 = 0$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \frac{10 \pm \sqrt{64}}{6}$$

$$= \frac{10 \pm 8}{6}$$

$x = 3 \text{ or } \frac{1}{3}$

c $5x^2 + 3x - 2 = 0$

$$x = \frac{-3 \pm \sqrt{9 + 40}}{10}$$

$$x = \frac{-3 \pm \sqrt{49}}{10}$$

$$x = \frac{-3 \pm 7}{10}$$

$x = -1 \text{ or } \frac{2}{5}$

d $21x^2 + 5x - 6 = 0$

$$x = \frac{-5 \pm \sqrt{25 + 504}}{42}$$

$$x = \frac{-5 \pm \sqrt{529}}{42}$$

$$x = \frac{-5 \pm 23}{42}$$

$x = -\frac{2}{3} \text{ or } \frac{3}{7}$

e $9x^2 - 6x + 35 = 0$

$$x = \frac{6 \pm \sqrt{36 - 1260}}{18}$$

no real roots

f $122x = 143x^2 + 24$
 $143x^2 - 122x + 24 = 0$

$$x = \frac{122 \pm \sqrt{14884 - 13728}}{286}$$

$$= \frac{122 \pm \sqrt{1156}}{286}$$

$$= \frac{122 \pm 34}{286}$$

$$x = \frac{4}{13} \text{ or } \frac{6}{11}$$

3 a $x^2 + 4x + 2 = 0$

$$x = \frac{-4 \pm \sqrt{16-8}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{2}$$

$$x = -2 \pm \sqrt{2}$$

b $5x^2 - 6x - 1 = 0$

$$x = \frac{6 \pm \sqrt{36+20}}{10}$$

$$= \frac{6 \pm \sqrt{56}}{10}$$

$$= \frac{6 \pm 2\sqrt{14}}{10}$$

$$x = \frac{3 \pm \sqrt{14}}{5}$$

c $3x^2 - x - 3 = 0$

$$x = \frac{1 \pm \sqrt{1+36}}{6}$$

$$x = \frac{1 \pm \sqrt{37}}{6}$$

d $2x^2 + 11x + 13 = 0$

$$x = \frac{-11 \pm \sqrt{121-104}}{4}$$

$$x = \frac{-11 \pm \sqrt{17}}{4}$$

e $11x^2 = 23x - 7$
 $11x^2 - 23x + 7 = 0$

$$x = \frac{23 \pm \sqrt{529-308}}{22}$$

$$= \frac{23 \pm \sqrt{221}}{22}$$

f $29x = 5x^2 - 41$
 $15x^2 - 29x - 41 = 0$

$$x = \frac{29 \pm \sqrt{841+820}}{10}$$

$$x = \frac{29 \pm \sqrt{1661}}{10}$$

4 a $x^2 + px - 2p^2 = 0$
 $(x + 2p)(x - p) = 0$
 $x = -2p \text{ or } p$

b $kx^2 + (k+2)x - 2 = 0$
 $(kx+2)(x+1) = 0$
 $x = \frac{-2}{k} \text{ or } -1$

c $2ax^2 + 6 = ax + 12x$
 $2ax^2 - ax - 12x + 6 = 0$
 $(ax-6)(2x-1) = 0$
 $x = \frac{6}{a} \text{ or } \frac{1}{2}$

d $x^2 - 2a^2 = b^2 - ax - 3ab$
 $x^2 + ax + (3ab - 2a^2 - b^2) = 0$

$$x = \frac{-a \pm \sqrt{a^2 - 12ab + 8a^2 + 4b^2}}{2}$$

$$x = \frac{-a \pm \sqrt{9a^2 - 12ab + 4b^2}}{2}$$

$$x = \frac{-a \pm (3a - 2b)}{2}$$
 $x = -2a + b \text{ or } a - b$

Exercise 3B

1 a $x^2 - 2x - 3 = 0$
 $\Delta = 4 - 4(1)(-3)$
 $\Delta = 16 > 0$
 $\therefore 2 \text{ real roots}$

b $x^2 + 10x + 25 = 0$
 $\Delta = 100 - 4(1)(25)$
 $\Delta = 0$
 $\therefore \text{one real root}$

c $4x^2 - 3x + 2 = 0$
 $\Delta = 9 - 4(4)(2)$
 $\Delta = -23 < 0$
 $\therefore \text{no real roots}$

d $5x^2 - 11x + 6 = 0$
 $\Delta = 121 - 4(5)(6)$
 $\Delta = 1 > 0$
 $\therefore 2 \text{ real roots}$

e $\frac{3}{5}x^2 - \frac{4}{7}x + \frac{2}{3} = 0$
 $\Delta = \frac{16}{49} - 4\left(\frac{3}{5}\right)\left(\frac{2}{3}\right)$
 $\Delta = -\frac{312}{245} < 0$
 $\therefore \text{no real roots}$

f $2x^2 + 2\sqrt{26}x + 13 = 0$
 $\Delta = 104 - 4(2)(13)$
 $\Delta = 0$
 $\therefore \text{one real root}$

2 a $x^2 - 2x - k = 0$

$$\Delta = 4 + 4k$$

$$4 + 4k = 0$$

$$k = -1$$

b $kx^2 + 3x - 2 = 0$

$$\Delta = 9 + 8k$$

$$9 + 8k > 0$$

$$k > -\frac{9}{8}$$

c $3x^2 + 5x + 2k - 1 = 0$

$$\Delta = 25 - 12(2k - 1)$$

$$= 37 - 24k$$

$$37 - 24k < 0$$

$$37 < 24k$$

$$k > \frac{37}{24}$$

d $x^2 - (3k + 2)x + k^2 = 0$

$$\Delta = (3k + 2)^2 - 4k^2$$

$$= 5k^2 + 12k + 4$$

$$5k^2 + 12k + 4 = 0$$

$$(5k + 2)(k + 2) = 0$$

$$k = -\frac{2}{5} \text{ or } -2$$

e $kx^2 + 2kx + k - 2 = 0$

$$\Delta = 4k^2 - 4k(k - 2) = 8k$$

$$8k > 0$$

$$k > 0$$

f $2kx^2 + (4k + 3)x + k - 3 = 0$

$$\Delta = (4k + 3)^2 - 8k(k - 3)$$

$$= 16k^2 + 24k + 9 - 8k^2 + 24k$$

$$= 8k^2 + 48k + 9$$

$$8k^2 + 48k + 9 < 0$$

if $8k^2 + 48k + 9 = 0$, then

$$k = \frac{-48 \pm \sqrt{2304 - 288}}{16}$$

$$= \frac{-48 \pm 12\sqrt{14}}{16}$$

$$= \frac{-12 \pm 3\sqrt{14}}{4}$$

$$\frac{-12 - 3\sqrt{14}}{4} < k < \frac{-12 + 3\sqrt{14}}{4}$$

Exercise 3C

1 a $x^2 - 3x + 2 = 0$

$$x_1 + x_2 = 3$$

$$x_1 x_2 = 2$$

$$\frac{2}{x_1} + \frac{2}{x_2} = \frac{2(x_2 + x_1)}{x_1 x_2} = 3$$

b $3x^2 - 5x + 1 = 0$

$$x_1 + x_2 = \frac{5}{3} \quad x_1 x_2 = \frac{1}{3}$$

$$3x_1^2 + 3x_2^2 = 3[(x_1 + x_2)^2 - 2x_1 x_2]$$

$$= 3\left[\frac{25}{9} - \frac{2}{3}\right]$$

$$= \frac{19}{3}$$

c $5x^2 + x + 3 = 0 \quad x_1 + x_2 = -\frac{1}{5} \quad x_1 x_2 = \frac{3}{5}$

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} = \frac{x_2^2 + x_1^2}{x_1^2 x_2^2} = \frac{(x_1 + x_2)^2 - 2x_1 x_2}{(x_1 x_2)^2}$$

$$= \frac{\frac{1}{25} - \frac{6}{5}}{\frac{9}{25}} = -\frac{29}{9}$$

d $x^2 - 2x + 4 = 0 \quad x_1 + x_2 = 2 \quad x_1 x_2 = 4$

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2$$

$$= 4 - 16 = -12$$

e $2x^2 - 4x + 3 = 0 \quad x_1 + x_2 = 2 \quad x_1 x_2 = \frac{3}{2}$

$$x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1 x_2 (x_1 + x_2)$$

$$= 8 - \frac{9}{2}(2) = -1$$

f $x^2 + 3x + 1 = 0 \quad x_1 + x_2 = -3 \quad x_1 x_2 = 1$

$$\frac{1}{x_1^4} + \frac{1}{x_2^4} = \frac{x_2^4 + x_1^4}{x_1^4 x_2^4}$$

$$(x_1 + x_2)^4 = x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + x_2^4$$

$$x_1^4 + x_2^4 = (x_1 + x_2)^4 - 6x_1^2 x_2^2 - 4x_1 x_2 (x_1^2 + x_2^2)$$

$$= (x_1 + x_2)^4 - 6x_1^2 x_2^2 - 4x_1 x_2 [(x_1 + x_2)^2 - 2x_1 x_2]$$

$$= 81 - 6 - 4(9 - 2)$$

$$= 47$$

$$\therefore \frac{1}{x_1^4} + \frac{1}{x_2^4} = \frac{47}{1^4} = 47$$

g $4x^2 - 7x + 1 = 0 \quad x_1 + x_2 = \frac{7}{4} \quad x_1 x_2 = \frac{1}{4}$

$$x_1^3 x_2^2 + x_1^2 x_2^3 = x_1^2 x_2^2 (x_1 + x_2)$$

$$= \frac{1}{16} \left(\frac{7}{4} \right)$$

$$= \frac{7}{64}$$

h $7x^2 + 4x - 5 = 0 \quad x_1 + x_2 = -\frac{4}{7} \quad x_1 x_2 = -\frac{5}{7}$

$$(x_1 - x_2)^4 = x_1^4 - 4x_1^3 x_2 + 6x_1^2 x_2^2 - 4x_1 x_2^3 + x_2^4$$

$$(x_1 + x_2)^4 = x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + x_2^4$$

$$\therefore (x_1 - x_2)^4 = (x_1 + x_2)^4 - 8x_1^3 x_2 - 8x_1 x_2^3$$

$$= (x_1 + x_2)^4 - 8x_1 x_2 (x_1^2 + x_2^2)$$

$$= (x_1 + x_2)^4 - 8x_1 x_2 [(x_1 + x_2)^2 - 2x_1 x_2]$$

$$= \left(-\frac{4}{7} \right)^4 + \frac{40}{7} \left(\frac{16}{49} + \frac{10}{7} \right)$$

$$= \frac{256}{2401} + \frac{40}{7} \left(\frac{86}{49} \right) = \frac{24336}{2401}$$

Exercise 3D

- 1 a $z = 3i$ $\operatorname{Re}(z) = 0$ $\operatorname{Im}(z) = 3$
 b $z = -7$ $\operatorname{Re}(z) = -7$ $\operatorname{Im}(z) = 0$
 c $z = \frac{18-12i}{8}$ $\operatorname{Re}(z) = \frac{18}{8} = \frac{9}{4}$ $\operatorname{Im}(z) = -\frac{12}{8} = -\frac{3}{2}$
 d $z = \frac{11}{4} + i\frac{\sqrt{7}}{5}$ $\operatorname{Re}(z) = \frac{11}{4}$ $\operatorname{Im}(z) = \frac{\sqrt{7}}{5}$
 e $z = \frac{4i-2}{3\pi^2}$ $\operatorname{Re}(z) = \frac{-2}{3\pi^2}$ $\operatorname{Im}(z) = \frac{4}{3\pi^2}$
- 2 a $|12+5i| = \sqrt{144+25} = \sqrt{169} = 13$
 b $|-24-7i| = \sqrt{576+49} = \sqrt{625} = 25$
 c $|2\sqrt{2} + i\sqrt{5}| = \sqrt{8+5} = \sqrt{13}$
 d $\left| \frac{-21+20i}{29} \right| = \sqrt{\frac{441}{841} + \frac{400}{841}} = 1$
 e $\left| \frac{-3+4i}{\pi} \right| = \sqrt{\frac{9}{\pi^2} + \frac{16}{\pi^2}} = \frac{5}{\pi}$

Exercise 3E

- 1 $z_1 = 2 + 3i$, $z_2 = \frac{3}{2} - 4i$, $z_3 = 1 - 5i$, $z_4 = \frac{3+4i}{5}$
- a $z_1 + z_3 = 3 - 2i$
 b $z_1 - 2z_3 = 2 + 3i - 2\left(\frac{3}{2} - 4i\right) = -1 + 11i$
 c $z_2 + z_4 = \frac{21}{10} - \frac{16}{5}i$
 d $5z_4 - 2z_2 = 5\left(\frac{3+4i}{5}\right) - 2\left(\frac{3}{2} - 4i\right) = 12i$
 e $3z_1 + 4z_2 - z_3 - 5z_4$
 $= 6 + 9i + 6 - 16i - 1 + 5i - 3 - 4i$
 $= 8 - 6i$
 f $z_1 \cdot z_2 - z_3 \cdot z_4 = (2+3i)\left(\frac{3}{2} - 4i\right) - (1-5i)\left(\frac{3+4i}{5}\right)$
 $= 3 - 8i + \frac{9}{2}i - 12i^2 - \frac{1}{5}(3+4i-15i-20i^2)$
 $= 15 - \frac{7}{2}i - \frac{23}{5} + \frac{11}{5}i$
 $= \frac{52}{5} - \frac{13}{10}i$
 g $z_3^2 - \frac{2}{3}(z_2 \cdot z_4) = (1-5i)^2 - \frac{2}{3}\left(\frac{3}{2} - 4i\right)\left(\frac{3+4i}{5}\right)$
 $= 1 - 10i + 25i^2 - \frac{2}{15}\left(\frac{9}{2} + 6i - 12i - 16i^2\right)$
 $= -24 - 10i - \frac{2}{15}\left(\frac{41}{2} - 6i\right)$
 $= -\frac{401}{15} - \frac{46i}{5}$

Exercise 3F

- 1 $z_1 = 1 + 4i$, $z_2 = 2 - i$, $z_3 = \frac{1}{2} - \frac{5}{2}i$, $z_4 = \frac{2i-1}{3}$
- a $\frac{z_1}{z_2} = \frac{1+4i}{2-i} \times \frac{2+i}{2+i} = \frac{2+i+8i-4}{4+1} = \frac{-2+9i}{5}$
 b $\frac{z_1^*}{z_2} = \frac{1-4i}{1+4i} \times \frac{1-4i}{1-4i} = \frac{1-4i-4i-16}{1+16} = \frac{-15-8i}{17}$
 c $\frac{z_2 \cdot z_4}{z_3} = \frac{(2-i)\frac{(2i-1)}{3}}{\frac{1}{2} - \frac{5}{2}i} = \frac{2}{3} \frac{(4i-2+2+i)}{1-5i}$
 $= \frac{2}{3} \times \frac{5i}{(1-5i)} \times \frac{1+5i}{1+5i} = \frac{2}{3} \left(\frac{5i-25}{1+25} \right) = \frac{-50+10i}{78}$
 $= \frac{-25}{39} + \frac{5}{39}i$
 d $\frac{3z_1 - 2z_3}{z_2 + 3z_4} = \frac{3+12i-1+5i}{2-i+2i-1} = \frac{2+17i}{1+i} \times \frac{1-i}{1-i}$
 $= \frac{2-2i+17i+17}{1+1}$
 $= \frac{19}{2} + \frac{15}{2}i$
 e $\frac{z_1^2}{(z_2^*)^2} = \frac{(1+4i)^2}{(2+i)^2} = \frac{1+8i-16}{4+4i-1} = \frac{-15+8i}{3+4i} \times \frac{3-4i}{3-4i}$
 $= \frac{-45+60i+24i+32}{9+16}$
 $= \frac{-13}{25} + \frac{84}{25}i$
- 2 a $(2+i)(a+ib) = 11-2i$
 $a+ib = \frac{11-2i}{2+i} \times \frac{2-i}{2-i}$
 $= \frac{22-11i-4i-2}{4+1}$
 $= 4-3i$ $a=4$, $b=-3$
 b $\frac{a+ib}{2-5i} = -3+2i$
 $a+ib = (-3+2i)(2-5i)$
 $= -6+15i+4i+10$
 $= 4+19i$
 $a=4$, $b=19$
 c $(3i-2)(a+ib) = 3+28i$
 $a+ib = \frac{3+28i}{-2+3i} \times \frac{-2-3i}{-2-3i} = \frac{-6-9i-56i+84}{4+9}$
 $= 6-5i$
 $a=6$, $b=-5$
 d $\left(\frac{1}{2} + \frac{3}{4}i\right)(a+ib) = -3+2i$
 $a+ib = \frac{-12+8i}{2+3i} \times \frac{2-3i}{2-3i} = \frac{-24+36i+16i+24}{4+9}$
 $= 4i$
 $a=0$, $b=4$

$$3 \quad a \quad \frac{3-2i}{4} \quad \operatorname{Re}(z) = \frac{3}{4} \quad \operatorname{Im}(z) = -\frac{1}{2}$$

$$b \quad \frac{5i-2}{3i} \times \frac{-3i}{-3i} = \frac{15+6i}{9} \quad \operatorname{Re}(z) = \frac{5}{3} \quad \operatorname{Im}(z) = \frac{2}{3}$$

$$c \quad \frac{1}{3i} + \frac{2}{1+i} = \frac{1+i+6i}{3i(1+i)} = \frac{1+7i}{-3+3i} \times \frac{-3-3i}{-3-3i}$$

$$= \frac{-3-3i-21+21}{9+9} = 1 - \frac{4}{3}i$$

$$\operatorname{Re}(z) = 1 \quad \operatorname{Im}(z) = -\frac{4}{3}i$$

$$d \quad \frac{2-3i}{2+3i} - \frac{2+3i}{2-3i} = \frac{(2-3i)^2 - (2+3i)^2}{4+9}$$

$$= \frac{4-12i-9-4-12i+9}{13} = \frac{-24i}{13}$$

$$\operatorname{Re}(z) = 0 \quad \operatorname{Im}(z) = \frac{-24}{13}$$

$$4 \quad z_1 = 1 + 3i \quad z_2 = 3 - i$$

$$a \quad z_1 \cdot z_2 + z_1 \cdot z_2^* = z_1(z_2 + z_2^*)$$

$$= (1 + 3i)(3 - i + 3 + i)$$

$$= 6 + 18i$$

$$b \quad z_1 \cdot z_2 - z_1^* \cdot z_2 = (z_1 - z_1^*) \cdot z_2$$

$$= (1 + 3i - (1 - 3i))(3 - i)$$

$$= 6i(3 - i)$$

$$= 6 + 18i$$

$$c \quad z_1 \cdot z_2 + (z_1 \cdot z_2)^* = (1 + 3i)(3 - i) + [(1 + 3i)(3 - i)]^*$$

$$= (6 + 8i) + (6 - 8i)$$

$$= 12$$

$$5 \quad a \quad (z + 1)i = (z + 2i)(3 + 2i)$$

$$zi + i = 3z + 2zi + 6i - 4$$

$$i - 6i + 4 = 3z + zi$$

$$z = \frac{4-5i}{3+i} \times \frac{3-i}{3-i} = \frac{12-4i-15i-5}{9+1}$$

$$= \frac{7}{10} - \frac{19}{10}i$$

$$b \quad (2z - 1)(1 + i) = (z - 1)(2 + 3i)$$

$$2z + 2zi - 1 - i = 2z + 3zi - 2 - 3i$$

$$\therefore 1 + 2i = zi$$

$$z = \frac{1+2i}{i} \times \frac{-i}{-i}$$

$$z = 2 - i$$

$$c \quad \frac{z-3i+2}{4+3i} = \frac{z-1}{1+i}$$

$$(z-3i+2)(1+i) = (z-1)(4+3i)$$

$$z + zi - 3i + 3 + 2 + 2i = 4z + 3zi - 4 - 3i$$

$$z + zi - i + 5 = 4z + 3zi - 4 - 3i$$

$$9 + 2i = 3z + 2zi$$

$$z = \frac{9+2i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{27-18i+6i+4}{9+4}$$

$$z = \frac{31}{13} - \frac{12}{13}i$$

$$d \quad \frac{38-2i}{2+i} = \frac{28+5}{10+15i}$$

$$(3z - 2i)(10 + 15i) = (2z + 5)(2 + i)$$

$$30z + 45zi - 20i + 30 = 4z + 2zi + 10 + 5i$$

$$26z + 43zi = -20 + 25i$$

$$z = \frac{-20+25i}{26+43i} \times \frac{26-43i}{26-43i} = \frac{-520+860i+650i+1075}{26^2+43^2}$$

$$= \frac{555+1510i}{2525} = \frac{111}{505} + \frac{302}{505}i$$

$$6 \quad \frac{z}{2-7i} \in \mathbb{R}. \quad \text{Let } z = a + bi$$

$$\frac{a+bi}{2-7i} \times \frac{2+7i}{2+7i} = \frac{2a+7ai+2bi-7b}{4+49}$$

$$\operatorname{Im}\left(\frac{z}{2-7i}\right) = 0 \quad \therefore 7a + 2b = 0$$

$$\therefore 7\operatorname{Re}(z) + 2\operatorname{Im}(z) = 0$$

$$7 \quad \text{Let } z = a + bi, \quad z^* = a - bi$$

$$\frac{3-5i}{z^*} = \frac{3-5i}{a-bi} \times \frac{a+bi}{a+bi} = \frac{3a+3bi-5ai-5b}{a^2+b^2}$$

$$\operatorname{Re}\left(\frac{3-5i}{z^*}\right) = 0 \quad \therefore 3a + 2b = 0$$

$$\therefore 3\operatorname{Re}(z) + 5\operatorname{Im}(z) = 0$$

$$8 \quad a \quad |z| - z = 4 + 3i. \quad \text{Let } z = a + ib$$

$$\sqrt{a^2+b^2} = 4 + 3i + a + ib$$

$$= (4+a) + i(3+b)$$

equating real and imaginary parts,

$$\sqrt{a^2+b^2} = 4 + a$$

$$3 + b = 0 \quad \therefore b = -3$$

$$a^2 + 9 = (4 + a)^2$$

$$a^2 + 9 = 16 + 8a + a^2$$

$$8a = -7$$

$$a = -\frac{7}{8} \quad \therefore z = -\frac{7}{8} - 3i$$

$$b \quad |z| + iz = 2 - i \quad \text{Let } z = a + ib$$

$$\sqrt{a^2+b^2} = 2 - i - i(a + ib)$$

$$= 2 - i - ia + b$$

$$= (2+b) - (1+a)i$$

$$\sqrt{a^2+b^2} = 2 + b \quad 1 + a = 0$$

$$a = -1$$

$$1 + b^2 = (2 + b)^2$$

$$1 + b^2 = 4 + 4b + b^2 \quad 4b = -3$$

$$b = -\frac{3}{4} \quad \therefore z = -1 - \frac{3}{4}i$$

$$c \quad z^2 - z^* = 0. \quad \text{Let } z = a + ib$$

$$(a + ib)^2 - (a - ib) = 0$$

$$a^2 + 2abi - b^2 - a + ib = 0$$

$$a^2 - b^2 - a = 0 \quad 2ab + b = 0$$

$$b(2a + 1) = 0$$

$$b = 0 \text{ or } a = -\frac{1}{2}$$

If $b = 0$, $a^2 - a = 0$, $a(a - 1) = 0$, $a = 0$ or 1

If $a = -\frac{1}{2}$, $\frac{1}{4} - b^2 + \frac{1}{2} = 0$, $b^2 = \frac{3}{4}$, $b = \pm \frac{\sqrt{3}}{2}$

$z = 0$, $z = 1$, $z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$, $z = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$

9 Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

$$\begin{aligned} \mathbf{a} \quad z_1 \cdot z_2 &= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \\ |z_1 \cdot z_2| &= \end{aligned}$$

$$\sqrt{(a_1a_2)^2 - 2a_1a_2b_1b_2 + (b_1b_2)^2 + (a_1b_2)^2 + 2a_1b_2a_2b_1 + (a_2b_1)^2}$$

$$= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2}$$

$$|z_1| \cdot |z_2| = \sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}$$

$$= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$= \sqrt{a_1^2a_2^2 + a_1^2b_2^2 + b_1^2a_2^2 + b_1^2b_2^2}$$

$$\therefore |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\mathbf{b} \quad \frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\left| \frac{z_1}{z_2} \right| = \sqrt{\frac{(a_1a_2 + b_1b_2)^2}{(a_2^2 + b_2^2)^2} + \frac{(a_2b_1 - a_1b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$= \frac{\sqrt{a_1^2a_2^2 + 2a_1a_2b_1b_2 + b_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 + a_1^2b_2^2}}{a_2^2 + b_2^2}$$

$$= \frac{\sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_2^2}}{a_2^2 + b_2^2}$$

$$= \frac{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}{a_2^2 + b_2^2} = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} = \frac{|z_1|}{|z_2|}$$

$$\therefore \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$$

$\mathbf{c}$ Let $p(n)$ be $|z^n| = |z|^n$

Step 1: When $n = 1$, $|z^n| = |z| = |z|^1$ so $p(1)$ is true.

Step 2: Assume $p(n)$ is true when $n = k$, i.e. $|z^k| = |z|^k$.

Step 3: Prove $p(n)$ is true when $n = k + 1$.

$$\begin{aligned} \text{Proof : } |z^{k+1}| &= |z^k \cdot z| \\ &= |z^k| |z| \\ &= |z|^k |z| \\ &= |z|^{k+1} \end{aligned}$$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by mathematical induction $p(n)$ is true.

$\mathbf{d}$ Let $z_1 = a + bi$ and $z_2 = c + di$.

Then $z_1 + z_2 = (a + c) + (b + d)i$.

From the properties of a triangle, $|z_1 + z_2|$

$$\leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2|.$$

10 Let $z = a + bi$, $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$

$$\mathbf{a} \quad z^* = a - bi \quad (z^*)^* = a + bi = z$$

$$\therefore (z^*)^* = z$$

$$\mathbf{b} \quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$(z_1 + z_2)^* = (a_1 + a_2) - i(b_1 + b_2)$$

$$= (a_1 - ib_1) + (a_2 - ib_2) = z_1^* + z_2^*$$

$$\therefore (z_1 + z_2)^* = z_1^* + z_2^*$$

$$\mathbf{c} \quad z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$$(z_1 \cdot z_2)^* = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$$

$$z_1^* \cdot z_2^* = (a_1 - ib_1)(a_2 - ib_2)$$

$$= (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$$

$$\therefore (z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$$

$$\mathbf{d} \quad z \cdot z^* = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$$

$$\therefore z \cdot z^* = |z|^2$$

$$\mathbf{e} \quad |z^*| = |a - ib| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

$$\therefore |z^*| = |z|$$

Exercise 3G

$$\mathbf{1} \quad \mathbf{a} \quad i^5 + i^8 + i^{14} + i^{19} = i + 1 - 1 - i = 0$$

$$\mathbf{b} \quad i^{123} + i^{172} + i^{256} + i^{375} = -i + 1 + 1 - i = 2 - 2i$$

$$\begin{aligned} \mathbf{c} \quad (2 - i^{53})(3 + 2i^{89}) &= (2 - 1)(3 + 2i) \\ &= 6 + 4i - 3i + 2 \\ &= 8 + i \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \frac{4i^{2010} - 3i^{2011}}{2i^{2012} + 5i^{2013}} &= \frac{-4 + 3i}{2 + 5i} \\ &= \frac{-4 + 3i}{2 + 5i} \times \frac{2 - 5i}{2 - 5i} = \frac{-8 + 20i + 6i + 15}{4 + 25} \\ &= \frac{7}{29} + \frac{26i}{29} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \frac{i + i^2 + i^3 + \dots + i^{2011}}{i \cdot i^2 \cdot i^3 \dots i^{2011}} &= \frac{i(1 - i^{2011})}{i^{(1+2+3+\dots+2011)}} \\ &= \frac{i(1 + i)}{(1 - i)i^{\frac{2011(2012)}{2}}} = \frac{i(1 + i)}{(1 - i)i^{2023066}} \\ &= \frac{i(1 + i)}{(1 - i)(-1)} = \frac{-1 + i}{-1 + i} = 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{i^2 + i^4 + i^6 + \dots + i^{2010}}{i^2 \cdot i^4 \cdot i^6 \dots i^{2010}} &= \frac{-1 + 1 - 1 + 1 \dots - 1}{i^{(2+4+6+\dots+2010)}} \\ &= \frac{-1}{i^{\frac{1005(2012)}{2}}} = \frac{-1}{i^{1011030}} = \frac{-1}{-1} = 1 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad (2 + 3i)^2 + (1 - 4i)^2 = 4 + 12i - 9 + 1 - 8i - 16 = -20 + 4i$$

$$\mathbf{b} \quad (3 + 2i)^2 + (3 - 2i)^2 = 9 + 12i - 4 + 9 - 12i - 4 = 10$$

$$\begin{aligned} \text{c } (3 + 2i)^3 &= 3^3 + 3(3)^2(2i) + 3(3)(2i)^2 + (2i)^3 \\ &= 27 + 54i - 36 - 8i \\ &= -9 + 46i \\ (3 - 2i)^3 &= 3^3 + 3(3)^2(-2i) + 3(3)(-2i)^2 + (-2i)^3 \\ &= 27 - 54i - 36 + 8i \\ &= -9 - 46i \\ \therefore (3 + 2i)^3 + (3 - 2i)^3 &= -18 \end{aligned}$$

$$\begin{aligned} \text{d } (1 + i)^4 &= (1 + i)^2(1 + i)^2 = (1 + 2i - 1)(1 + 2i - 1) = -4 \\ (1 - i)^4 &= (1 - i)^2(1 - i)^2 = (1 - 2i - 1)(1 - 2i - 1) = -4 \\ \therefore (1 + i)^4 + (1 - i)^4 &= -8 \end{aligned}$$

$$\begin{aligned} \text{3 a } \sqrt{3 + 4i} &= x + iy \\ 3 + 4i &= x^2 - y^2 + 2ixy \\ x^2 - y^2 &= 3 \quad 2xy = 4 \\ y &= \frac{2}{x} \\ x^2 - \frac{4}{x^2} &= 3 \quad \therefore x^4 - 3x^2 - 4 = 0 \\ (x^2 - 4)(x^2 + 1) &= 0 \\ \therefore x &= \pm 2 \end{aligned}$$

$$\sqrt{3 + 4i} = 2 + i \text{ or } -2 - i$$

$$\begin{aligned} \text{b } \sqrt{12i - 5} &= x + iy \\ 12i - 5 &= x^2 - y^2 + 2ixy \\ x^2 - y^2 &= -5 \quad 2xy = 12 \\ y &= \frac{6}{x} \\ x^2 - \frac{36}{x^2} &= -5 \quad \therefore x^4 + 5x^2 - 36 = 0 \\ (x^2 + 9)(x^2 - 4) &= 0 \\ x &= \pm 2 \end{aligned}$$

$$\sqrt{12i - 5} = 2 + 3i \text{ or } -2 - 3i$$

$$\begin{aligned} \text{c } \sqrt{\frac{5}{4} + 3i} &= x + iy \\ \frac{5}{4} + 3i &= x^2 - y^2 + 2ixy \\ x^2 - y^2 &= \frac{5}{4}, \quad 2xy = 3 \\ \therefore y &= \frac{3}{2x} \\ x^2 - \frac{9}{4x^2} &= \frac{5}{4} \\ 4x^4 - 5x^2 - 9 &= 0 \\ (4x^2 - 9)(x^2 + 1) &= 0 \\ x &= \pm \frac{3}{2} \end{aligned}$$

$$\sqrt{\frac{5}{4} + 3i} = \frac{3}{2} + i \text{ or } -\frac{3}{2} - i$$

$$\begin{aligned} \text{d } \sqrt{\frac{55}{144} - \frac{1}{3}i} &= x + iy \\ \frac{55}{144} - \frac{1}{3}i &= x^2 - y^2 + 2ixy \\ x^2 - y^2 &= \frac{55}{144} \quad 2xy = -\frac{1}{3} \\ y &= -\frac{1}{6x} \end{aligned}$$

$$x^2 - \frac{1}{36x^2} = \frac{55}{144}$$

$$36x^4 - \frac{55}{4}x^2 - 1 = 0 \quad \therefore 144x^4 - 55x^2 - 4 = 0$$

$$(9x^2 - 4)(16x^2 + 1) = 0$$

$$x = \pm \frac{2}{3}$$

$$x = \frac{2}{3}, y = -\frac{1}{4} \quad x = -\frac{2}{3}, y = \frac{1}{4}$$

$$\sqrt{\frac{55}{144} - \frac{1}{3}i} = -\frac{2}{3} + \frac{1}{4}i \text{ or } \frac{2}{3} - \frac{1}{4}i$$

$$\text{e } \sqrt{i} = x + iy$$

$$i = x^2 - y^2 + 2ixy$$

$$x^2 = y^2 \quad 2xy = 1$$

$$y = \frac{1}{2x}$$

$$x^2 = \frac{1}{4x^2}, x^4 = \frac{1}{4} \quad x = \pm \frac{1}{\sqrt{2}} \quad y = \pm \frac{1}{\left(\frac{2}{\sqrt{2}}\right)} = \pm \frac{1}{\sqrt{2}}$$

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \text{ or } -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

$$\text{f } \sqrt{-i} = x + iy$$

$$-i = x^2 - y^2 + 2ixy$$

$$x^2 = y^2 \quad 2xy = -1$$

$$y = -\frac{1}{2x}$$

$$\text{As in e, } x = \pm \frac{1}{\sqrt{2}}, \sqrt{-i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \text{ or } -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\begin{aligned} \text{4 a } (1 + i)^{2n} &= [(1 + i)^2]^n = [1 + 2i - 1]^n \\ \therefore (1 + i)^{2n} &= (2i)^n \end{aligned}$$

$$\begin{aligned} \text{b } (1 + i)^{2n+1} &= (1 + i)(1 + i)^{2n} = (1 + i)(2i)^n \\ \text{from a } \therefore (1 + i)^{2n+1} &= (1 + i)(2i)^n \end{aligned}$$

Exercise 3H

$$\text{1 } f(x) = 2x^2 + 3x + 1 \quad g(x) = 3x^2 - 2x - 5$$

$$\begin{aligned} \text{a } \lambda \cdot f(x) + \mu \cdot g(x) &= 2\lambda x^2 + 3\lambda x + \lambda + 3\mu x^2 \\ &\quad - 2\mu x - 5\mu \\ &= (2\lambda + 3\mu)x^2 + (3\lambda - 2\mu)x + (\lambda - 5\mu) \end{aligned}$$

$$2\lambda + 3\mu = 0$$

$$3\lambda - 2\mu = 13$$

$$\lambda - 5\mu = 13$$

$$\lambda = 3, \mu = -2$$

$$\text{b } 2\lambda + 3\mu = 26$$

$$3\lambda - 2\mu = 26$$

$$\lambda - 5\mu = 0$$

$$\lambda = 10, \mu = 2$$

$$\text{2 a}$$

	x^3	$-2x$
x^2	x^5	$-2x^3$
2	$2x^3$	$-4x$

$$f(x) \cdot g(x) = x^5 - 4x$$

b	$27x^3$	$-36x^2$	$48x$	-64
$3x^2$	$81x^5$	$-108x^4$	$144x^3$	$-192x^2$
$7x$	$189x^4$	$-252x^3$	$336x^2$	$-448x$
4	$108x^3$	$-144x^2$	$192x$	-256

$$f(x) \cdot g(x) = 81x^5 + 81x^4 - 256x - 256$$

$$\begin{aligned} 3 \quad f(x) \cdot g(x) &= (ax^2 - 3x + 5)(7x^2 + bx - 3) \\ &= 7ax^4 + abx^3 - 3ax^2 - 21x^3 - 3bx^2 + 9x \\ &\quad + 35x^2 + 5bx - 15 \\ &= 7ax^4 + (ab - 21)x^3 + (-3a - 3b + 35)x^2 \\ &\quad + (9 + 5b)x - 15 \end{aligned}$$

$$7a = 14$$

$$ab - 21 = -17$$

$$-3a - 3b + 35 = 23$$

$$9 + 5b = 19 \quad a = 2 \quad b = 2$$

$$\begin{aligned} 4 \quad f(x) \cdot g(x) &= (x^3 + ax^2 - x + 2)(2x^2 + bx + c) \\ &= 2x^5 + bx^4 + cx^3 + 2ax^4 + abx^3 \\ &\quad + acx^2 - 2x^3 - bx^2 - cx + 4x^2 + 2bx + 2c \end{aligned}$$

$$2a + b = -5$$

$$c + ab - 2 = 3$$

$$ac - b + 4 = 5$$

$$-c + 2b = -8$$

$$a = -1 \quad b = -3 \quad c = 2$$

$$\begin{aligned} 5 \quad (x^2 + px + q)^2 &= (x^2 + px + q)(x^2 + px + q) \\ &= x^4 + px^3 + qx^2 + px^3 + p^2x^2 + pqx + qx^2 \\ &\quad + pxq + q^2 \\ &= x^4 + 2px^3 + (2q + p^2)x^2 + 2pqx + q^2 \\ 2p &= 6 \quad p = 3 \\ 2q + p^2 &= a \quad q = \pm 2 \\ 2pq &= b \quad \text{If } q = 2, \quad a = 13, \quad b = 12 \\ q^2 &= 4 \quad \text{If } q = -2, \quad a = 5, \quad b = -12 \\ a = 13, b = 12, f(x) &= (x^2 + 3x + 2)^2 \\ \text{or } a = 5, b = -12, f(x) &= (x^2 + 3x - 2)^2 \end{aligned}$$

$$\begin{aligned} 6 \quad f(x) &= x^3 + 12x^2 + 6x + 3 \\ g(x) &= f(x - 2) \\ &= (x - 2)^3 + 12(x - 2)^2 + 6(x - 2) + 3 \\ &= x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3 + \\ &\quad 12(x^2 - 4x + 4) + 6x - 12 + 3 \\ &= x^3 - 6x^2 + 12x - 8 + 12x^2 - 48x + 48 + 6x - 9 \\ g(x) &= x^3 + 6x^2 - 30x + 31 \end{aligned}$$

$$\begin{aligned} 7 \quad f(2x - 1) &= 16x^4 - 32x^3 + 12x^2 \\ \text{Let } y &= 2x - 1, \quad x = \frac{y+1}{2} \\ f(y) &= 16 \frac{(y+1)^4}{16} - 32 \frac{(y+1)^3}{8} + 12 \frac{(y+1)^2}{4} \end{aligned}$$

$$\begin{aligned} &= y^4 + 4y^3 + 6y^2 + 4y + 1 - 4(y^3 + 3y^2 + 3y + 1) \\ &\quad + 3(y^2 + 2y + 1) \\ f(y) &= y^4 - 3y^2 - 2y \\ \therefore f(x) &= x^4 - 3x^2 - 2x \end{aligned}$$

$$\begin{aligned} 8 \quad f(0) &= 4 \Rightarrow e = 4 \\ f(10) &= 32584 \Rightarrow 10000a + 1000b + 100c + 10d + 4 \\ &= 32584 \\ \text{Since } a, b, c, d &\in \mathbb{Z}^+ \text{ and are less than } 10, \\ a = 3, b = 2, c = 5, d = 8 \\ \Rightarrow f(x) &= 3x^4 + 2x^3 + 5x^2 + 8x + 4 \end{aligned}$$

Exercise 3I

$$\begin{array}{r} 1 \quad a \quad x+2 \overline{) \begin{array}{r} x^3+3x^2+2x-1 \\ x^4+5x^3+8x^2+3x-2 \\ \hline x^4+2x^3 \\ \hline 3x^3+8x^2 \\ 3x^3+6x^2 \\ \hline 2x^2+3x \\ 2x^2+4x \\ \hline -x-2 \\ -x-2 \\ \hline \end{array}} \end{array}$$

$$q(x) = x^3 + 3x^2 + 2x - 1$$

$$\begin{array}{r} b \quad x^2+0x-1 \overline{) \begin{array}{r} x^3+3x^2+2x-1 \\ x^5+3x^4+x^3-4x^2-2x+1 \\ \hline x^5+0x^4-x^3 \\ \hline 3x^4+2x^3-4x^2 \\ 3x^4+0x^3-3x^2 \\ \hline 2x^3-x^2-2x \\ 2x^3+0x^2-2x \\ \hline -x^2+0x+1 \\ -x^2+0x+1 \\ \hline \end{array}} \end{array}$$

$$q(x) = x^3 + 3x^2 + 2x - 1$$

$$\begin{array}{r} c \quad x^2+x+1 \overline{) \begin{array}{r} 2x^3-5x^2+4x-1 \\ 2x^5-3x^4+x^3-2x^2+3x-1 \\ \hline 2x^5+2x^4+2x^3 \\ \hline -5x^4-x^3-2x^2 \\ -5x^4-5x^3-5x^2 \\ \hline -4x^3+3x^2+3x \\ 4x^3+4x^2+4x \\ \hline -x^2-x-1 \\ -x^2-x-1 \\ \hline \end{array}} \end{array}$$

$$q(x) = 2x^3 - 5x^2 + 4x - 1$$

$$\begin{array}{r}
 2x^3 + 3x^2 + x + 3 \\
 x+1 \overline{) 2x^4 + 5x^3 + 4x^2 + 4x + 3} \\
 \underline{2x^4 + 2x^3} \\
 3x^3 + 4x^2 \\
 \underline{3x^3 + 3x^2} \\
 x^2 + 4x \\
 \underline{x^2 + x} \\
 3x + 3 \\
 \underline{3x + 3} \\
 0
 \end{array}$$

$$q(x) = 2x^3 + 3x^2 + x + 3 \quad r(x) = 0$$

$$\begin{array}{r}
 3x^2 - 2x + 1 \\
 x^2 + 2x + 3 \overline{) 3x^4 + 4x^3 + 6x^2 - 2x + 6} \\
 \underline{3x^4 + 6x^3 + 9x^2} \\
 -2x^3 - 3x^2 - 2x \\
 \underline{-2x^3 - 4x^2 - 6x} \\
 x^2 + 4x + 6 \\
 \underline{x^2 + 2x + 3} \\
 2x + 3
 \end{array}$$

$$q(x) = 3x^2 - 2x + 1 \quad r(x) = 2x + 3$$

$$\begin{array}{r}
 x^4 - x^3 + x - 1 \\
 x^2 + x + 1 \overline{) x^6 + 0x^5 + 0x^4 + 0x^3 + 0x^2 + x - 1} \\
 \underline{x^6 + x^5 + x^4} \\
 -x^5 - x^4 + 0x^3 \\
 \underline{-x^5 - x^4 - x^3} \\
 x^3 + 0x^2 + x \\
 \underline{x^3 + x^2 + x} \\
 -x^2 + 0x - 1 \\
 \underline{-x^2 - x - 1} \\
 x
 \end{array}$$

$$q(x) = x^4 - x^3 + x - 1 \quad r(x) = x$$

Exercise 3J

- 1 a $x^3 - x^2 - 4x - 5 = (x^2 + 2x + 2)(x - 3) + 1$,
 $q(x) = x^2 + 2x + 2 \quad r(x) = 1$
- b $2x^3 + 5x^2 + 4x + 3 = (2x^2 + 3x + 1)(x + 1) + 2$,
 $q(x) = 2x^2 + 3x + 1 \quad r(x) = 2$
- c $x^5 - 3x^3 - 2x + 1 = (x^4 - 2x^3 + x^2 - 2x + 2)(x + 2) - 3$,
 $q(x) = x^4 - 2x^3 + 2x^2 - 2x + 2, \quad r(x) = -3$
- d $3x^6 - 2x^4 + 5x^2 - 2 = (3x^5 + 3x^4 + x^3 + x^2 + 6x + 6)(x - 1) + 4$,
 $q(x) = 3x^5 + 3x^4 + x^3 + x^2 + 6x + 6, \quad r(x) = 4$

- 2 $f(2) = 4 \times 16 - 27 \times 4 + 25 \times 2 - 6 = 0$
 $\therefore (x - 2)$ is a factor.
 $f(-3) = 4 \times 81 - 27 \times 9 - 25 \times 3 - 6 = 0$
 $\therefore (x + 3)$ is a factor.

Exercise 3K

- 1 a $q(x) = x^4 - x^3 + x^2 + 2x + 1 \quad r(x) = -2$
b $q(x) = x^3 + x^2 + x - 1 \quad r(x) = 7$
- 2 $f(x) = (x^2 + 2x - 1)(3x - 4) + x + 2$
 $= 3x^3 - 4x^2 + 6x^2 - 8x - 3x + 4 + x + 2$
 $f(x) = 3x^3 + 2x^2 - 10x + 6$
- 3 $f(x) = x^5 - 4x^4 + 3x^3 + 2x^2 - 3x + a$
 $f(3) = 0 \quad 243 - 324 + 81 + 18 - 9 + a = 0$
 $a = -9$
- 4 $f(x) = x^5 - 2x^4 + 2x^3 + bx - 1$
 $f(1) = 0 \quad 1 - 2 + 2 + b - 1 = 0$
 $\therefore b = 0$
- 5 $f(x) = 4x^3 + 5x^2 + ax + b$
 $f(-2) = 0 \quad f(1) = 6$
 $-32 + 20 - 2a + b = 0 \quad 4 + 5 + a + b = 6$
 $-2a + b = 12 \quad a + b = -3$
 $a = -5, b = 2$
- 6 $f(x) = (x^2 - 2x - 3)q(x) + ax + b$
 $f(x) = (x - 3)(x + 1)q(x) + ax + b$
 $f(3) = 2 \quad \therefore 2 = 3a + b$
 $f(-1) = -4 \quad \therefore -4 = -a + b$
 $\therefore a = 1.5 \quad b = -2.5$
 $\therefore \text{remainder} = \frac{3}{2}x - \frac{5}{2}$
- 7 $f(x) = x^{2011} + x^{2010} + \dots + x + 1 = 0$
remainder $= f(-1) = -1 + 1 - 1 + \dots - 1 + 1 = 0$
 $\therefore \text{remainder} = 0$
- 8 $f(x) = (x + 1)^{2n} + (x + 2)^n - 1$
 $f(-1) = 0^{2n} + (1)^n - 1 = 0 + 1 - 1 = 0$
 $\therefore f(x)$ is divisible by $(x + 1)$
 $f(-2) = (-1)^{2n} + 0^n - 1 = 1 + 0 - 1 = 0$
 $\therefore f(x)$ is divisible by $(x + 2)$
 $(x + 1)(x + 2) = x^2 + 3x + 2$
 $\therefore f(x)$ is divisible by $x^2 + 3x + 2$
- 9 $f(x) = (ax - b)q(x) + r(x)$
 $f\left(\frac{b}{a}\right) = \left(\frac{ab}{a} - b\right)q\left(\frac{b}{a}\right) + \text{remainder}$
 $= (0)q\left(\frac{b}{a}\right) + \text{remainder}$
 $\therefore \text{remainder} = f\left(\frac{b}{a}\right)$

Exercise 3L

- 1 a** $f(x) = 2x^4 + 3x^3 - 10x^2 - 12x + 8$
 $= (x + 2)^2 g(x)$
 $= (x^2 + 4x + 4)(2x^2 - 5x + 2)$
 $= (x + 2)(x + 2)(2x - 1)(x - 2)$
- b** $f(x) = 12x^3 - 32x^2 + 23x - 5$
 $= (2x - 1)^2 g(x)$
 $= (4x^2 - 4x + 1)(3x - 5)$
 $= (2x - 1)^2(3x - 5)$
- 2 a** $f(x) = x(x - 1)(x - 3)(x - 5)$
 $= (x - 1)(x^2 - 8x + 15)$
 $= x^3 - 9x^2 + 23x - 15$
- b** $f(x) = x(x + 2)(x + 1)(x - 1)$
 $= x(x + 2)(x^2 - 1)$
 $= x(x^3 + 2x^2 - x - 2)$
 $= x^4 + 2x^3 - x^2 - 2x$
- c** $f(x) = (3x + 2)(x - 1)(x - 2)(x - 3)$
 $= (3x^2 - x - 2)(x^2 - 5x + 6)$
 $= 3x^4 - 16x^3 + 21x^2 + 4x - 12$
- 3 a** $f(x) = (x^2 - 2)(x^2 - 3) = x^4 - 5x^2 + 6$
- b** $f(x) = (2x + 1)(4x - 3)(x^2 - 5)$
 $= (2x + 1)(4x^3 - 3x^2 - 20x + 15)$
 $= 8x^4 - 2x^3 - 43x^2 + 10x + 15$
- c** $f(x) = (5x + 3)(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2}))(x^3 - 3)$
 $= (5x + 3)(x^2 - 2x - 1)(x^3 - 3)$
 $= (5x + 3)(x^5 - 2x^4 - x^3 - 3x^2 + 6x + 3)$
 $= 5x^6 - 7x^5 - 11x^4 - 18x^3 + 21x^2 + 33x + 9$
- 4 a** $f(x) = x^3 - 2x^2 - 5x + 6$
 $= (x - 1)(x^2 - x - 6)$
 $= (x - 1)(x - 3)(x + 2)$
- b** $f(x) = 2x^3 - x^2 - 7x + 6$
 $= (x - 1)(2x^2 + x - 6)$
 $= (x - 1)(2x - 3)(x + 2)$
- c** $f(x) = 5x^4 - 12x^3 - 14x^2 + 12x + 9$
 $= (x - 1)(x + 1)g(x)$
 $= (x^2 - 1)(5x^2 - 12x - 9)$
 $= (x - 1)(x + 1)(5x + 3)(x - 3)$

Exercise 3M

- 1 a** $(x - 2i)(x + 2i) = x^2 + 4$
 $f(x) = x^3 + 3x^2 + 4x + 12$
 $= (x^2 + 4)(x + 3)$
 remaining zeros are $-2i$ and -3

- b** $(x - (1 - 2i))(x - (1 + 2i)) = x^2 - 2x + 5$
 $f(x) = x^3 - 6x^2 + 13x - 20$
 $= (x^2 - 2x + 5)(x - 4)$
 remaining zeros are $1 + 2i$ and 4
- c** $\left(x - \left(-\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{3}{2} - \frac{\sqrt{3}}{2}i\right)\right) = x^2 + 3x + 3$
 $f(x) = 5x^3 + 17x^2 + 21x + 6$
 $= (x^2 + 3x + 3)(5x + 2)$
 remaining zeros are $-\frac{3}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{2}{5}$
- d** $(x - i)(x + i) = x^2 + 1$
 $f(x) = x^4 - 6x^3 + 5x^2 - 4x + 4$
 $= (x^2 + 1)(x^2 - 4x + 4)$
 $= (x + i)(x - i)(x - 2)^2$
 remaining zeros are i , 2 and 2
- e** $(x - (-1 - 3i))(x - (-1 + 3i)) = x^2 + 2x + 10$
 $f(x) = 2x^4 + 3x^3 + 17x^2 - 12x - 10$
 $= (x^2 + 2x + 10)(2x^2 - x - 1)$
 $= (x - (-1 - 3i))(x - (-1 + 3i))(2x + 1)(x - 1)$
 remaining zeros are $-1 + 3i$, $-\frac{1}{2}$, 1
- f** $(x - (-2 + i))(x - (-2 - i)) = x^2 + 4x + 5$
 $f(x) = 2x^4 + 9x^3 + 11x^2 - 7x - 15$
 $= (x^2 + 4x + 5)(2x^2 + x - 3)$
 $= (x - (-2 + i))(x - (-2 - i))(2x + 3)(x - 1)$
 remaining zeros are $-2 - i$, $-\frac{3}{2}$, 1
- g** $\left(x - \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}i\right)\right) = x^2 + x + \frac{3}{2}$
 $f(x) = 6x^4 + 26x^3 + 35x^2 + 36x + 9$
 $= (2x^2 + 2x + 3)(3x^2 + 10x + 3)$
 $= 2\left(x - \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}i\right)\right)(3x + 1)(x + 3)$
 remaining zeros are $-\frac{1}{2} - \frac{\sqrt{5}}{2}i$, $-\frac{1}{3}$, -3
- h** $\left(x - \left(\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)\right)\left(x - \left(\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)\right) = x^2 - \frac{2}{3}x + \frac{1}{3}$
 $f(x) = 3x^4 - 2x^3 + 4x^2 - 2x + 1$
 $= (3x^2 - 2x + 1)(x^2 + 1)$
 remaining zeros are $\frac{1}{3} - \frac{\sqrt{2}}{3}i$, i , $-i$
- 2 a** $f(-1) = 0$
 $\therefore -1 + 13 + a = 0$
 $\therefore a = -12$
 $f(x) = x^3 - 13x - 12$
 $= (x + 1)(x^2 - x - 12)$
 $= (x + 1)(x - 4)(x + 3)$
 remaining zeros are -3 and 4

- b** $f(3) = 0 \quad \therefore 27 - 63 + 3a - 15 = 0$
 $\therefore a = 17$
 $f(x) = x^3 - 7x^2 + 17x - 15$
 $= (x - 3)(x^2 - 4x + 5)$
 remaining zeros are $\frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$
- c** $f(-1 - i) = 0, (-1 - i)^4 + 2(-1 - i)^3$
 $- 2(-1 - i)^2 - 8(-1 - i) + a = 0$
 $-4 + 2(2 - i) - 2(2i) - 8(-1 - i) + a = 0$
 $-4 + 4 - 4i - 4i + 8 + 8i + a = 0$
 $\therefore a = -8$
 $f(x) = x^4 + 2x^3 - 2x^2 - 8x - 8$
 $(x - (-1 - i))(x - (-1 + i)) = x^2 + 2x + 2$
 $f(x) = (x^2 + 2x + 2)(x^2 - 4)$
 $= (x - (-1 - i))(x - (-1 + i))(x - 2)(x + 2)$
 remaining zeros are $-1 + i, 2, -2$
- d** $f(-2i) = 0 \quad (-2i)^4 - 4(-2i)^3 + 9(-2i)^2 +$
 $a(-2i) + b = 0$
 $16 - 32i - 36 - 2ai + b = 0$
 $-32 - 2a = 0 \quad -20 + b = 0$
 $a = -16 \quad b = 20$
 $f(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$
 $(x - 2i)(x + 2i) = x^2 + 4$
 $f(x) = (x^2 + 4)(x^2 - 4x + 5)$
 remaining zeros are $2i, 2 + i, 2 - i$

Exercise 3N

- 1** $3x^3 - 2x^2 - 5x - 4 = 0$
- a** $x_1 + x_2 + x_3 = \frac{2}{3}$
- b** $x_1 \cdot x_2 \cdot x_3 = \frac{4}{3}$
- c** $x_1x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = -\frac{5}{3}$
- d** $\frac{6}{x_1} + \frac{6}{x_2} + \frac{6}{x_3} = \frac{6(x_2x_3 + x_1x_3 + x_1x_2)}{x_1x_2x_3} = \frac{6(-\frac{5}{3})}{\frac{4}{3}} = -\frac{15}{2}$
- e** $9x_1^2 + 9x_2^2 + 9x_3^2 = 9[(x_1 + x_2 + x_3)^2$
 $- 2(x_1x_2 + x_1x_3 + x_2x_3)]$
 $= 9\left[\left(\frac{2}{3}\right)^2 - 2\left(-\frac{5}{3}\right)\right] = 9\left(\frac{4}{9} + \frac{10}{3}\right)$
 $= 34$
- 2** $x^4 - 3x^3 + 2x^2 - 4x - 6 = 0$
- a** $x_1 + x_2 + x_3 + x_4 = 3$
- b** $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = -6$
- c** $x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_3 \cdot x_4 = 2$
- d** $x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4 = 4$

- e** $\frac{3}{x_1} + \frac{3}{x_2} + \frac{3}{x_3} + \frac{3}{x_4} = \frac{3(4)}{-6} = -2$
- f** $(x_1 + x_2 + x_3 + x_4)^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 +$
 $2(x_1 \cdot x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$
 $\therefore \frac{x_1^2}{5} + \frac{x_2^2}{5} + \frac{x_3^2}{5} + \frac{x_4^2}{5} = \frac{1}{5}[3^2 - 2(2)] = 1$
- 3 a** $f(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$
 sum $= -\frac{2}{1} = -2$ product $= (-1)^4 \frac{5}{1} = 5$
- b** $f(x) = 4x^6 + x^5 + 7x^4 - 3x^3 + 2x$
 sum $= -\frac{1}{4}$ product $= 0$
- c** $f(x) = 11x^{10} - \frac{3}{7}x^7 + \sqrt{5}x^3 - \pi x + 22$
 sum $= 0$ product $= (-1)^{10} \frac{22}{11} = 2$
- d** $f(x) = 5x^{7007} - 4x^{7006} + 2x^{231} + 10x + 8$
 sum $= \frac{4}{5}$ product $= -\frac{8}{5}$

Exercise 3O

- 1 a** $x^3 - 6x^2 + 11x - 6 = 0$
 $(x - 1)(x^2 - 5x + 6) = 0$
 $(x - 1)(x - 2)(x - 3) = 0$
 $x = 1, 2 \text{ or } 3$
- b** $x^3 + 2x^2 - 7x + 4 = 0$
 $(x - 1)(x^2 + 3x - 4) = 0$
 $(x - 1)(x - 1)(x + 4) = 0$
 $x = 1, 1, -4$
- c** $x^3 + 3x^2 - 4x - 12 = 0$
 $(x - 2)(x^2 + 5x + 6) = 0$
 $(x - 2)(x + 2)(x + 3) = 0$
 $x = 2, -2, -3$
- d** $2x^3 - 5x^2 - 18x + 45 = 0$
 $(x - 3)(2x^2 + x - 15) = 0$
 $(x - 3)(2x - 5)(x + 3) = 0$
 $x = 3, \frac{5}{2}, -3$

Exercise 3P

- 1 a** $12x^3 + 17x^2 + 2x - 3 = 0$
 $(x + 1)(12x^2 + 5x - 3) = 0$
 $(x - 1)(3x - 1)(4x + 3) = 0$
 $x = -1, \frac{-3}{4}, \frac{1}{3}$
- b** $x^3 - 4x^2 - 5x + 14 = 0$
 $(x + 2)(x^2 - 6x + 7) = 0$
 $x = -2 \text{ or } x = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm \sqrt{8}}{2} = 3 \pm \sqrt{2}$
 $x = -2, 3 \pm \sqrt{2}$

c $3x^3 - 13x^2 + 11x + 14 = 0$

$(3x + 2)(x^2 - 5x + 7) = 0$

$x = \frac{-2}{3}$

d $x^4 - x^3 - 11x^2 + 9x + 18 = 0$

$(x + 1)(x - 2)g(x) = 0$

$(x^2 - x - 2)(x^2 - 9) = 0$

$(x + 1)(x - 2)(x - 3)(x + 3) = 0$

$x = -1, 2, 3, -3$

2 $x^3 + ax^2 - x - 3 = 0$

a $-27 + 9a + 3 - 3 = 0 \quad \therefore 9a = 27$

$a = 3$

b $x^3 + 3x^2 - x - 3 = 0$

$(x + 3)(x^2 - 1) = 0$

$(x + 3)(x - 1)(x + 1) = 0$

other roots are 1, -1

3 $ax^3 - 7x^2 + bx + 4 = 0$

a $(x - 2)(x - 2)(ax + 1) = 0$

$(x^2 - 4x + 4)(ax + 1) = 0$

$ax^3 + x^2 - 4ax^2 - 4x + 4ax + 4 = 0$

$ax^3 + (1 - 4a)x^2 + (4a - 4)x + 4 = 0$

$1 - 4a = -7 \quad \therefore a = 2$

$4a - 4 = b \quad b = 4$

b remaining root $= -\frac{1}{a} = -\frac{1}{2}$

4 Let $x = a$ be an integer zero

$\therefore a^3 + 5a + p = 0$

$p = -a(a^2 + 5)$

$\therefore a$ and $a^2 + 5$ are factors of p

$\therefore p$ is not prime

$\therefore$ If p is prime there are no integer zeros

5 $f(x) = x^3 + ax^2 + bx + c$

a Let the 2 zeros be $p, -p$

$f(x) = (x - p)(x + p) \left(x - \frac{c}{p^2} \right)$

$= (x^2 - p^2) \left(x - \frac{c}{p^2} \right)$

$= x^3 - \frac{c}{p^2}x^2 - p^2x + c$

$\therefore a = -\frac{c}{p^2} \quad b = -p^2$

$\therefore ab = \left(-\frac{c}{p^2} \right) (-p^2) = c \quad \therefore ab = c$

b third zero $= \frac{c}{p^2} = \frac{-c}{b} = -a$

Exercise 3Q

1 a $x^3 - 6x^2 + 11x - 6 \geq 0$

$(x - 1)(x^2 - 5x + 6) \geq 0$

$(x - 1)(x - 2)(x - 3) \geq 0$

x	$]-\infty, 1[$	$]1, 2[$	$]2, 3[$	$]3, \infty[$
$x - 1$	-	+	+	+
$x - 2$	-	-	+	+
$x - 3$	-	-	-	+
$f(x)$	-	+	-	+

$\therefore x \in [1, 2] \cup [3, \infty[$

b $x^3 + 2x^2 - 7x + 4 \leq 0$

$(x - 1)(x^2 + 3x - 4) \leq 0$

$(x - 1)(x - 1)(x + 4) \leq 0$

x	$]-\infty, -4[$	$]-4, 1[$	$]1, \infty[$
$(x - 1)^2$	+	+	+
$(x + 4)$	-	+	+
$f(x)$	-	+	+

$\therefore x \in]-\infty, -4] \text{ or } x = 1$

c $x^3 + 3x^2 - 4x - 12 < 0$

$(x - 2)(x^2 + 5x + 6) < 0$

$(x - 2)(x + 2)(x + 3) < 0$

x	$]-\infty, -3[$	$]-3, -2[$	$] -2, 2[$	$]3, \infty[$
$x - 2$	-	-	-	+
$x + 2$	-	-	+	+
$x + 3$	-	+	+	+
$f(x)$	-	+	-	+

$\therefore x \in]-\infty, -3] \cup] -2, 2[$

d $2x^3 - 5x^2 - 18x + 45 > 0$

$(x - 3)(2x^2 + x - 15) > 0$

$(x - 3)(2x - 5)(x + 3) > 0$

x	$]-\infty, -3[$	$]-3, 2.5[$	$]2.5, 3[$	$]3, \infty[$
$x - 3$	-	-	-	+
$2x - 5$	-	-	+	+
$x + 3$	-	+	+	+
$f(x)$	-	+	-	+

$\therefore x \in]-3, 2.5] \cup]3, \infty[$

e $12x^3 + 17x^2 + 2x - 3 \leq 0$

$(x + 1)(12x^2 + 5x - 3) \leq 0$

$(x + 1)(4x + 3)(3x - 1) \leq 0$

x	$]-\infty, -1[$	$]-1, -\frac{3}{4}[$	$]-\frac{3}{4}, \frac{1}{3}[$	$]\frac{1}{3}, \infty[$
$x + 1$	-	+	+	+
$4x + 3$	-	-	+	+
$3x - 1$	-	-	-	+
$f(x)$	-	+	-	+

$\therefore x \in]-\infty, -1] \cup [\frac{-3}{4}, \frac{1}{3}]$

$$\begin{aligned} \text{f } x^3 - 4x^2 - 5x + 14 &> 0 \\ (x+2)(x^2 - 6x + 7) &> 0 \quad \frac{6 \pm \sqrt{36-28}}{2} = 3 \pm \sqrt{2} \end{aligned}$$

x	$]-\infty, -2[$	$]-2, 3-\sqrt{2}[$	$]3-\sqrt{2}, 3+\sqrt{2}[$	$]3+\sqrt{2}, \infty[$
$x+2$	-	+	+	+
$x-(3+\sqrt{2})$	-	-	-	+
$x-(3-\sqrt{2})$	-	-	+	+
$f(x)$	-	+	-	+

$$x \in]-2, 3-\sqrt{2}[\cup]3+\sqrt{2}, \infty[$$

$$\begin{aligned} \text{g } 3x^3 - 13x^2 + 11x + 14 &< 0 \\ (3x+2)(x^2 - 5x + 7) &< 0 \\ x &= \frac{-2}{3} \end{aligned}$$

x	$]-\infty, \frac{-2}{3}[$	$]\frac{-2}{3}, \infty[$
$3x+2$	-	+
$x^2 - 5x + 7$	+	+
$f(x)$	-	+

$$x \in]-\infty, \frac{-2}{3}[$$

$$\begin{aligned} \text{h } x^4 - x^3 - 11x^2 + 9x + 18 &\geq 0 \\ (x+1)(x-2)g(x) &\geq 0 \\ (x^2 - x - 2)(x^2 - 9) &\geq 0 \\ (x+1)(x-2)(x-3)(x+3) &\geq 0 \end{aligned}$$

x	$]-\infty, -3[$	$]-3, -1[$	$]-1, 2[$	$]2, 3[$	$]3, \infty[$
$x+1$	-	-	+	+	+
$x-2$	-	-	-	+	+
$x-3$	-	-	-	-	+
$x+3$	-	+	+	+	+
$f(x)$	+	-	+	-	+

$$\therefore x \in]-\infty, -3] \cup [-1, 2] \cup [3, \infty[$$

$$2 \quad f(x) > g(x)$$

$$\begin{aligned} 4x^3 - 17x^2 + 30x + 5 &> -2x^3 \\ + 8x^2 + 9x - 5 & \\ 6x^3 - 25x^2 + 21x + 10 &> 0 \\ (x-2)(6x^2 - 13x - 5) &> 0 \\ (x-2)(3x+1)(2x-5) &> 0 \end{aligned}$$

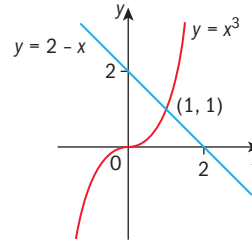
x	$]-\infty, -\frac{1}{3}[$	$]-\frac{1}{3}, 2[$	$]2, \frac{5}{2}[$	$]\frac{5}{2}, \infty[$
$x-1$	-	-	+	+
$3x+1$	-	+	+	+
$2x-5$	-	-	-	+
$f(x) - g(x)$	-	+	-	+

$$x \in]-\frac{1}{3}, 2[\cup]\frac{5}{2}, \infty[$$

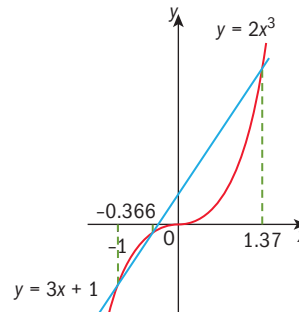
$$\begin{aligned} \text{3 a } x^7 - 2x^2 - 1 &\geq 0 \\ x &\in [-1, -0.921] \cup [1.26, \infty[\end{aligned}$$

$$\begin{aligned} \text{b } x^9 - 2x^8 + 2x^5 + x &\leq 0 \\ x &\in]-\infty, 0] \end{aligned}$$

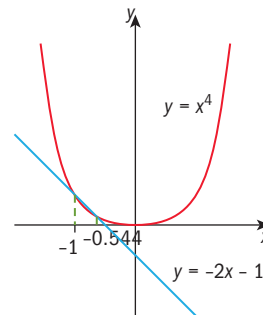
$$\begin{aligned} \text{4 a } x^3 + x - 2 &> 0 \\ x^3 &> 2 - x \\ x &\in]1, \infty[\end{aligned}$$



$$\begin{aligned} \text{b } -2x^3 + 3x + 1 &\geq 0 \\ 3x + 1 &\geq 2x^3 \\ x &\in]-\infty, -1] \cup [-0.366, 1.37] \end{aligned}$$



$$\begin{aligned} \text{c } x^4 + 2x + 1 &\leq 0 \\ x^4 &\leq -2x - 1 \\ x &\in [-1, -0.544] \end{aligned}$$



Exercise 3R

$$\begin{aligned} \text{1 a } 2ix + (2+3i)y &= i \quad a = 2i \quad b = 2+3i \quad e = i \\ (1+i)x + 2y &= 3 \quad c = 1+i \quad d = 2 \quad f = 3 \end{aligned}$$

$$\begin{aligned} D &= \begin{vmatrix} 2i & 2+3i \\ 1+i & 2 \end{vmatrix} = 4i - (1+i)(2+3i) \\ &= 4i - 2 - 5i + 3 = 1 - i \end{aligned}$$

$$D_x = \begin{vmatrix} i & 2+3i \\ 3 & 2 \end{vmatrix} = 2i - 3(2+3i) = -6 - 7i$$

$$D_y = \begin{vmatrix} 2i & i \\ 1+i & 3 \end{vmatrix} = 6i - i(1+i) = 5i + 1$$

$$x = \frac{-6-7i}{1-i} \times \frac{1+i}{1+i} = \frac{-6-6i-7i+7}{2} = \frac{1}{2} - \frac{13}{2}i$$

$$y = \frac{1+5i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+5i-5}{2} = -2 + 3i$$

b $(1+i)x + 3iy = 2 + 6i$ $a = 1+i$ $b = 3i$
 $e = 2 + 6i$
 $(2-i)x - (4+3i)y = 4i - 3$ $c = 2-i$
 $d = -4-3i$ $f = -3+4i$

$$D = \begin{vmatrix} 1+i & 3i \\ 2-i & -4-3i \end{vmatrix} = (1+i)(-4-3i) - 3i(2-i)$$

$$= 4 - 3i - 4i + 3 - 6i - 3 = -4 - 13i$$

$$D_x = \begin{vmatrix} 2+6i & 3i \\ -3+4i & -4-3i \end{vmatrix} = (2+6i)(-4-3i)$$

$$-3i(-3+4i)$$

$$= -8 - 6i - 24i + 18 + 9i + 12 = 22 - 21i$$

$$D_y = \begin{vmatrix} 1+i & 2+6i \\ 2-i & -3+4i \end{vmatrix} = (1+i)(-3+4i) - (2+6i)$$

$$(2-i)$$

$$= -3 + 4i - 3i - 4 - (4 - 2i + 12i + 6)$$

$$= -17 - 9i$$

$$x = \frac{22 - 21i}{-4 - 13i} \times \frac{-4 + 13i}{-4 + 13i} = \frac{-88 + 286i + 84i + 273}{185} = 1 + 2i$$

$$y = \frac{-17 - 9i}{-4 - 13i} \times \frac{-4 + 13i}{-4 + 13i} = \frac{68 - 221i + 36i + 117}{185} = 1 - i$$

2 a $x + y = -1$ (1) $x + y = -1$ (1)
 $x + z = 4$ (2) (2) - (3) $x - y = 3$ (4)
 $y + z = 1$ (3)

$$(1) + (4) \quad 2x = 2 \quad \therefore x = 1$$

$$\text{sub in (1)} \quad 1 + y = -1 \quad y = -2$$

$$\text{sub in (3)} \quad -2 + z = 1 \quad z = 3 \quad (1, -2, 3)$$

b $x - 5y + 3z = -1$ (1) (1) + 3(3) $7x - 2y = 5$ (4)

$$3x - y + 2z = 4$$
 (2) (2) + 2(3) $7x + y = 8$ (5)

$$2x + y - z = 2$$
 (3)

$$(5) - (4) \quad 3y = 3, y = 1$$

$$\text{sub in (5)} \quad 7x + 1 = 8 \quad x = 1$$

$$\text{sub in (3)} \quad 2 + 1 - z = 2 \quad z = 1 \quad (1, 1, 1)$$

c $2x + y + 2z = 0$ (1) 4(1) + (2):
 $14x + 3z = -2$ (4)

$$6x - 4y - 5z = -2$$
 (2) (3) - (1):
 $2x - 5z = 2$ (5)

$$4x + y - 3z = 2$$
 (3)

$$(4) - 7(5) \quad 38z = -16 \quad z = \frac{-8}{19}$$

$$\text{sub in (5)} \quad 2x + \frac{40}{19} = 2, \quad x = \frac{-1}{19}$$

$$\text{sub in (1)} \quad \frac{-2}{19} + y - \frac{16}{19} = 0 \quad y = \frac{18}{19} \quad \left(\frac{-1}{19}, \frac{18}{19}, \frac{-8}{19} \right)$$

d $3x - 4y + 3z = -2$ (1) (1) + (3) $5x - 10y = -10$
 $x + 2y + 6z = 6$ (2) $x - 2y = -2$ (4)
 $2x - 6y - 3z = -8$ (3) (2) + 2(3) $5x - 10y = -10$
 $x - 2y = -2$ (5)

(4) + (5) are the same equation $\therefore$ infinite number of solutions $x = 2y - 2$

$$\text{sub in (1)} \quad 6y - 6 - 4y + 3z = -2, \quad 3x = 4 - 2y,$$

$$z = \frac{4-2y}{3}$$

$$\left(2y - 2, y, \frac{4-2y}{3} \right)$$

e $x + 2y + z = 4$ (1) (2) - 2(1) $-3y = -3, y = 1$
 $2x + y + 2z = 5$ (2)

$$3x + 2y + 3z = 12$$
 (3)

$$\text{sub in (2)} \quad 2x + 2z = 4$$

$$\text{sub in (3)} \quad 3x + 3z = 10 \quad \therefore \text{no solution}$$

f $2x - 3y + 5z = -1$ (1) (1) - 2(3) $y - z = -19$ (4)
 $9x - 7y + 16z = 0$ (2) (2) - 9(3) $11y - 11z = -81$ (5)
 $x - 2y + 3z = 9$ (3)

(4) + (5) are inconsistent $\therefore$ no solution

3 a $x + 2y + z = 0$ (1) (1) - (3) $z - kz = -2$

$$2x + y + 2z = 1$$
 (2) $(1-k)z = -2$

$$x + 2y + kz = 2$$
 (3) no solution if $k = 1$

$$\text{If } k \neq 1, z = \frac{-2}{1-k}$$

$$\text{sub in (1)} \quad x + 2y = \frac{2}{1-k} \quad (4)$$

$$\text{sub in (2)} \quad 2x + y = 1 + \frac{4}{1-k} = \frac{5-k}{1-k} \quad (5)$$

$$(5) - 2(4) \quad -3y = \frac{5-k}{1-k} - \frac{4}{1-k}$$

$$-3y = \frac{1-k}{1-k} \quad \therefore y = -\frac{1}{3}$$

$$\text{sub in (4)} \quad x = \frac{2}{1-k} + \frac{2}{3} = \frac{6+2(1-k)}{3(1-k)} = \frac{8-2k}{3(1-k)}$$

$\therefore$ no unique solution if $k = 1$

b $x + y + z = 1$ (1) (2) - 2(1) $(k-2)y + z = -4$ (4)

$$2x + ky + 3z = -2$$
 (2) (3) - 3(1) $2y + (k-3)z = -4$ (5)

$$3x + 5y + kz = -1$$
 (3)

For no unique solution

$$\frac{k-2}{2} = \frac{1}{k-3}$$

$$(k-2)(k-3) = 2$$

$$k^2 - 5k + 4 = 0$$

$$(k-4)(k-1) = 0$$

$$k = 1 \text{ or } 4$$

4 a $x + 2y + 3z = 1$ (1) $(1) - (2) \quad (1 - k)x - 2y = -1$ (4)
 $kx + 4y + 3z = 2$ (2) $2(1) + 3(3) \quad 11x + 22y = 11$

$3x + 6y - 2z = 3$ (3) $x + 2y = 1$ (5)
 For an infinite number of solutions
 $1 - k = -1, k = 2$

$x = 1 - 2y$
 sub in (3) $2z = 3x + 6y - 3$
 $= 3(1 - 2y) + 6y - 3$
 $\therefore z = 0$
 $(1 - 2y, y, 0)$

b $x + y + z = 1$ (1)
 $2x + ky + 3z = -2$ (2)
 $3x + 5y + kz = -1$ (3)
 From **3b**, $(k - 2)y + z = -4$ (4)
 $2y + (k - 3)z = -4$ (5)
 For no unique solution $k = 1$ or 4
 If $k = 1$: $-y + z = -4$ } no solution
 $2y - 2z = -4$ }

If $k = 4$: $2y + z = -4$ } infinitely many solutions
 $2y + z = -4$ }
 $z = -4 - 2y$

sub in (1) $x = 1 - y - (-4 - 2y) = 5 + y$
 $(y + 5, y, -4 - 2y)$

5 $x + y + z = m$ (1) $(1) - (3) \quad z - mz = m + 1$
 $x + my + z = 2m$ (2) $z(1 - m) = m + 1$
 $x + y + mz = -1$ (3) unique solution if $m \neq 1$
 $z = \frac{1 + m}{1 - m}$

sub in (1) $x + y + \frac{1 + m}{1 - m} = m$

$x + y = m - \frac{1 + m}{1 - m} = \frac{m - m^2 - 1 - m}{1 - m}$

$x + y = \frac{-m^2 - 1}{1 - m}$ (4)

sub in (2) $x + my = 2m - \frac{1 + m}{1 - m} = \frac{2m - 2m^2 - 1 - m}{1 - m}$

$x + my = \frac{m - 2m^2 - 1}{1 - m}$ (5)

(4) - (5): $y(1 - m) = \frac{-m^2 - 1 - m + 2m^2 + 1}{1 - m}$

$\frac{m^2 - m}{1 - m} = \frac{m(m - 1)}{1 - m} = -m$

$\therefore y = \frac{-m}{1 - m}$

sub in (4) $x = \frac{-m^2 - 1}{1 - m} + \frac{m}{1 - m} = \frac{m - m^2 - 1}{1 - m}$

$\left(\frac{m - m^2 - 1}{1 - m}, \frac{-m}{1 - m}, \frac{1 + m}{1 - m} \right)$



Review exercise

1 $f(x) = x^4 - 3x^3 + ax^2 - 4x + 7$
 $f(-2) = 7 \quad \therefore 16 + 24 + 4a + 8 + 7 = 0$
 $4a = -55$
 $a = \frac{-55}{4}$

2 $3x - 2y = i - 2$
 $4y - (1 - i)x = 3 + 3i \Rightarrow -(1 - i)x + 4y = 3 + 3i$
 $a = 3 \quad b = -2 \quad c = -1 + i \quad d = 4 \quad e = -2 + i$
 $f = 3 + 3i$

$D = \begin{vmatrix} 3 & -2 \\ -1 + i & 4 \end{vmatrix} = 12 + 2(-1 + i) = 10 + 2i$

$D_x = \begin{vmatrix} -2 + i & -2 \\ 3 + 3i & 4 \end{vmatrix} = 4(-2 + i) + 2(3 + 3i) = -2 + 10i$

$D_y = \begin{vmatrix} 3 & -2 + i \\ -1 + i & 3 + 3i \end{vmatrix} = 3(3 + 3i) - (-1 + i)(-2 + i)$
 $= 9 + 9i - (2 - 3i - 1) = 8 + 12i$

$x = \frac{-2 + 10i}{10 + 2i} \times \frac{10 - 2i}{10 - 2i} = \frac{-20 + 4i + 100i + 20}{104} = i$

$y = \frac{8 + 12i}{10 + 2i} \times \frac{10 - 2i}{10 - 2i} = \frac{80 - 16x + 120i + 24}{104} = 1 + i$

$x = i, y = 1 + i$

3 $f(x) = m - 2 + (2m + 1)x + mx^2 \quad m < 0$
 $b^2 - 4ac < 0$
 $(2m + 1)^2 - 4(m - 2)m < 0$
 $4m^2 + 4m + 1 - 4m^2 + 8m < 0$
 $12m + 1 < 0$
 $m < \frac{-1}{12}$

4 $z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$
 $(z - (1 - 2i))(z - (1 + 2i)) = z^2 - 2z + 5$
 $(z^2 - 2z + 5)(z^2 + 9) = 0$
 remaining roots are $1 + 2i, -3i, 3i$

5 $mx + 2y = 1$
 $4x + (m + 2)y = 4$ no unique solution
 $\frac{m}{4} = \frac{2}{m + 2} \quad m^2 + 2m = 8$
 $m^2 + 2m - 8 = 0$
 $(m + 4)(m - 2) = 0$
 $m = -4$ or 2

6 $x^2 + ax + a + 1 = 0$
 $\alpha + \beta = -a$
 $\alpha\beta = a + 1$
 $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$$\begin{aligned}
 (\alpha + \beta)^3 &= (\alpha^3 + \beta^3) + 3\alpha\beta(\alpha + \beta) \\
 -a^3 &= 9 - 3a(a + 1) \\
 a^3 - 3a^2 - 3a + 9 &= 0 \\
 (a - 3)(a^2 - 3) &= 0 \\
 a &= 3, \pm\sqrt{3}
 \end{aligned}$$

7 $P_n: z^{2^n} = \frac{i^n}{2^n}, \quad n \in \mathbb{Z}^+$

Let $n = 1, z^2 = \frac{(1+i)^2}{2^2} = \frac{1+2i-1}{4} = \frac{i}{2}$

$\therefore P_1$ is true

Assume $P_k: z^{2^k} = \frac{i^k}{2^k}$

Prove $P_{k+1}: z^{2^{k+1}} = z^2 z^{2^k} = \frac{i}{2} \left(\frac{i^k}{2^k} \right) = \frac{i^{k+1}}{2^{k+1}}$

$\therefore P_k \Rightarrow P_{k+1}$

$\therefore$ by mathematical induction, $z^{2^n} = \frac{i^n}{2^n}$

8 $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i-1}{2} = i$

$$\left(\frac{1+i}{1-i} \right)^{2011} = i^{2011} = i^{2008} i^3 = -i$$

$\therefore$ imaginary part = -1

9 $x^3 - 5x^2 + 6x - 3 = 0$

$$\alpha + \beta + \gamma = 5$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = 6 \quad \alpha\beta\gamma = 3$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2}{\alpha^2\beta^2\gamma^2}$$

$$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 + 2(\alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha\beta\gamma^2)$$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha\beta\gamma(\beta + \alpha + \gamma)$$

$$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 = 6^2 + 2(3)(5) = 66$$

$$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{6}{3^2} = \frac{6}{9} = \frac{2}{3}$$

10 a Let $a = \sqrt[3]{7-\sqrt{50}}, b = \sqrt[3]{7+\sqrt{50}}, x = a + b$

$$x^3 + 3x - 14 = (a+b)^3 + 3(a+b) - 14$$

$$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b) - 14$$

$$= a^3 + b^3 + 3ab(a+b) + 3(a+b) - 14$$

$$a^3 + b^3 + 3(a+b)(ab+1) - 14$$

$$ab = \sqrt[3]{(7-\sqrt{50})(7+\sqrt{50})} = \sqrt[3]{49-50} = \sqrt[3]{-1} = -1$$

$$\therefore x^3 + 3x - 14 = a^3 + b^3 + 3(a+b)(-1+1) - 14$$

$$= a^3 + b^3 - 14 = 0$$

$$= 7 - \sqrt{50} + 7 + \sqrt{50} - 14 = 0$$

$$\therefore \sqrt[3]{7-\sqrt{50}} + \sqrt[3]{7+\sqrt{50}} \text{ satisfies the equation}$$

$$x^3 + 3x - 14$$

b $f(x) = z^3 + 3z - 14$

$$= (z-2)(z^2 + 2z + 7)$$

$$z = 2 \text{ or } z = \frac{-2 \pm \sqrt{4-28}}{2} = -1 \pm i\sqrt{6}$$

$$z = 2, -1 \pm i\sqrt{6}$$

c since 2 is the only real root,

$$\sqrt[3]{7-\sqrt{50}} + \sqrt[3]{7+\sqrt{50}} = 2$$



Review exercise

1 $x \in [1.67, \infty[$

2 $(mx)^2 + 3x + 1 - m = 0$

$$b^2 - 4ac < 0$$

$$9 - 4m^2(1 - m) < 0$$

$$9 - 4m^2 + 4m^3 < 0$$

$$m \in]-\infty, -1.05[$$

3 $2x + 14y + 9z = -7 \quad 2x + 14y + 9z = -7$

$$4x - 3z = 4 + 7y \quad 4x - 7y - 3z = 4$$

$$10x - 28y = 5 + 6z \quad 10x - 28y - 6z = 5$$

$$x = \frac{1}{2}, y = \frac{2}{7}, z = \frac{-4}{3}$$

4 $3x^3 + 2x = 5x^2 + 4$

$$3x^3 - 5x^2 + 2x - 4 = 0$$

$$\alpha + \beta + \gamma = \frac{5}{3}, \alpha\beta + \beta\gamma + \alpha\gamma = \frac{2}{3}, \alpha\beta\gamma = \frac{4}{3}$$

$$(\alpha + \beta + \gamma)^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma)$$

$$= \alpha^3 + \beta^3 + \gamma^3 + 6\alpha\beta\gamma + 3(\alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma)$$

$$\alpha^3 + \beta^3 + \gamma^3 = \frac{125}{27} - 6\left(\frac{4}{3}\right) - 3(\alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta +$$

$$\alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma)$$

$$(\alpha + \beta + \gamma)(\gamma\beta + \beta\gamma + \alpha\gamma) = 3\alpha\beta\gamma + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma$$

$$\therefore \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma$$

$$= \frac{5}{3}\left(\frac{2}{3}\right) - 3\left(\frac{4}{3}\right) = \frac{-26}{9}$$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = \frac{125}{27} - 6\left(\frac{4}{3}\right) - 3\left(\frac{-26}{9}\right)$$

$$= \frac{143}{27}$$

5 $f(x) = x^7 + 35x^6 - 97x^5 + 33x^2 + 4$

$$\text{smallest zero} = 0.833$$

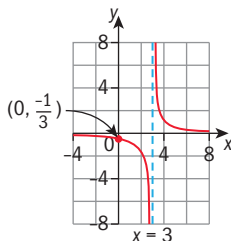
4

Modeling the real world

Answers

Skills check

1



$$2 \quad \sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{1-\frac{1}{2}} = 10$$

Exercise 4A

$$1 \quad \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = -2$$

$$2 \quad \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$$

$$3 \quad \lim_{x \rightarrow 2^+} f(x) = \frac{1}{3} \quad \lim_{x \rightarrow 2^-} f(x) = 5$$

$$\therefore \lim_{x \rightarrow 2} \begin{cases} 3x - 1 & x < 2 \\ \frac{1}{x^2 - 1} & x \geq 2 \end{cases} \text{ does not exist}$$

$$4 \quad \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \quad \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1 \quad \therefore \lim_{x \rightarrow 0} \frac{|x|}{x} \text{ does not exist}$$

$$5 \quad \lim_{x \rightarrow 6} (x - 6)^{\frac{2}{3}} = 0$$

$$6 \quad \lim_{x \rightarrow 3^-} [x] = 2 \quad \lim_{x \rightarrow 3^+} [x] = 3$$

$$\therefore \lim_{x \rightarrow 3} [x] \text{ does not exist.}$$

Exercise 4B

$$1 \quad \lim_{x \rightarrow 1^-} f(x) = 0 \quad \lim_{x \rightarrow 1^+} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) \text{ does not exist}$$

$\therefore f$ is not continuous at $x = 1$

$$2 \quad \lim_{x \rightarrow -2^-} f(x) = 1 \quad \lim_{x \rightarrow -2^+} f(x) = 1 \quad \therefore \lim_{x \rightarrow -2} f(x) = 1$$

Also, $f(-2) = 1 \quad \therefore f$ is continuous at $x = -2$

$$3 \quad \lim_{x \rightarrow 1^-} f(x) = -1 \quad \lim_{x \rightarrow 1^+} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 1} f(x) \text{ does not exist}$$

$$\therefore f \text{ is not continuous at } x = 1$$

$$4 \quad \lim_{x \rightarrow 3^-} f(x) = 8 \quad \therefore f(3) = 8 \quad \therefore 6 = 8 \quad \therefore k = \frac{4}{3}$$

$$5 \quad \lim_{x \rightarrow 3^-} f(x) = 4 \quad \therefore f(3) = 4 \quad \therefore 9a - a = 4$$

$$8a = 4$$

$$a = \frac{1}{2}$$

- 6 a discontinuous at $x = \pm 1$
 b discontinuous at $x = \pm 2$
 c continuous
 d discontinuous at $x = -4$ and $x = 1$
 e discontinuous at $x = 1$
 f continuous

Exercise 4C

$$1 \quad a \quad \lim_{x \rightarrow 4} \left(\frac{x+3}{x-3} \right) = 7$$

$$b \quad \lim_{x \rightarrow -2} \left(\frac{x^2 + x - 2}{x + 2} \right) = \lim_{x \rightarrow -2} \left(\frac{(x+2)(x-1)}{(x+2)} \right)$$

$$= \lim_{x \rightarrow -2} (x-1) = -3$$

$$c \quad \lim_{x \rightarrow -2} \left(\frac{x^6 - 64}{x^3 - 8} \right) = \lim_{x \rightarrow -2} \left(\frac{(x^3 + 8)(x^3 - 8)}{x^3 - 8} \right)$$

$$= \lim_{x \rightarrow -2} (x^3 + 8) = 0$$

$$d \quad \lim_{x \rightarrow 0} \left(\frac{x^2 - 1}{x^2 - x} \right) = \lim_{x \rightarrow 0} \left(\frac{(x-1)(x+1)}{x(x-1)} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{x+1}{x} \right) = \lim_{x \rightarrow 0} \left(1 + \frac{1}{x} \right),$$

which does not exist

$$e \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \rightarrow 1} \left(1 + \frac{1}{x} \right) = 2$$

$$f \quad \lim_{x \rightarrow 1} \frac{1}{1 + \frac{1}{1-x}} = \lim_{x \rightarrow 1} \frac{1-x}{2-x} = 0$$

$$g \quad \lim_{x \rightarrow 0} \frac{(2+3x)^2 - 4(1+x)^2}{6x}$$

$$= \lim_{x \rightarrow 0} \frac{4 + 12x + 9x^2 - 4 - 8x - 4x^2}{6x}$$

$$= \lim_{x \rightarrow 0} \frac{5x^2 + 4x}{6x} = \lim_{x \rightarrow 0} \frac{5x + 4}{6} = \frac{2}{3}$$

$$h \quad \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} = \lim_{x \rightarrow a} \frac{(x-a)(x+a)}{x-a}$$

$$= \lim_{x \rightarrow a} (x+a) = 2a$$

2 a $\lim_{x \rightarrow \infty} \frac{2x}{x+2} = 2$

b $\lim_{x \rightarrow \infty} \frac{3x^2}{x^2-1} = 3$

c $\lim_{x \rightarrow \infty} \frac{2x^2+x-1}{3x^2+5x-1} = \frac{2}{3}$

d $\lim_{x \rightarrow \infty} \frac{5x^2}{4x^3+2} = 0$

e $\lim_{x \rightarrow \infty} \frac{x-1}{x^2-3x+5} = 0$

f $\lim_{x \rightarrow \infty} \frac{\sqrt{4x+3}+2\sqrt{1+x}}{\sqrt{x}} = 4$

3 a $y = 3$ b $y = \frac{1}{2}$ c $y = 0$

d $y = -1$ e no horizontal asymptote

Exercise 4D

1 a converges b converges c converges
d diverges e converges

2 a converges, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

b $\sum_{n=1}^{\infty} \left(\frac{\pi}{3.14}\right)^n$ diverges since $\frac{\pi}{3.14} > 1$

c converges, $\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{5}{2}$

d converges, $\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{9} = \frac{1}{3}$

e converges, $\sum_{n=1}^{\infty} \frac{2^n-3^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n$
 $= \frac{\frac{2}{7}}{1-\frac{2}{7}} - \frac{\frac{3}{7}}{1-\frac{3}{7}} = \frac{2}{5} - \frac{3}{4} = \frac{-7}{20}$

f converges, $\sum_{n=1}^{\infty} 4(-0.6)^{n-1} = \frac{4}{1+0.6} = 2.5$

3 $u_1 = 35$ $r = 2^x$

a $-1 < 2^x < 1$ 2^x must be positive
 $\therefore 0 < 2^x < 1 \therefore x < 0$

b $\frac{35}{1-2^x} = 40 \therefore 1-2^x = \frac{7}{8} \therefore 2^x = \frac{1}{8} \therefore x = -3$

4 $-1 < \frac{3x}{x+1} < 1 \therefore -0.25 < x < 0.5$

Exercise 4E

1 a $y = 2x^2 - 1$ ($x = 1$)

$$\frac{\Delta y}{\Delta x} = \frac{[2(1+h)^2-1]-[2(1)^2-1]}{(1+h)-1} = \frac{(1+4h+2h^2)-1}{h}$$
$$= \frac{4h+2h^2}{h} = 4+2h$$

gradient = $\lim_{h \rightarrow 0} (4+2h) = 4$

b $y = \frac{2}{x}$ ($x = -2$)

$$\frac{\Delta y}{\Delta x} = \frac{\frac{2}{-2+h} - \frac{2}{-2}}{\frac{-2}{h} + 1} = \frac{2-2+h}{h(-2+h)} = \frac{1}{-2+h}$$

gradient = $\lim_{h \rightarrow 0} \left(\frac{1}{-2+h}\right) = -\frac{1}{2}$

c $y = x^3$ ($x = 1$)

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^3-1^3}{(1+h)-1} = \frac{1+3h+3h^2+h^3-1}{h} = 3+3h+h^2$$

gradient = $\lim_{h \rightarrow 0} (3+3h+h^2) = 3$

$\therefore$ gradient = 3

d $y = -x^2$ ($x = 1$)

$$f'(x) = \lim_{h \rightarrow 0} \frac{-(x+h)^2 - (-x^2)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{-x^2-2xh-h^2+x^2}{h}$$
$$= \lim_{h \rightarrow 0} (-2x-h)$$

$= -2x$

$f'(1) = -2(1) = -2$

e $y = \frac{x}{x+1}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h}$$
$$= \lim_{h \rightarrow 0} \frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)h}$$

$$= \lim_{h \rightarrow 0} \frac{h}{(x+h+1)(x+1)h} = \lim_{h \rightarrow 0} \frac{1}{(x+h+1)(x+1)}$$
$$= \frac{1}{(x+1)^2}$$

$f'(0) = 1$

f $y = \frac{1}{x^2}$ ($x = 2$)

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$
$$= \lim_{h \rightarrow 0} \frac{x^2 - x^2 - 2xh - h^2}{hx^2(x+h)^2} = \lim_{h \rightarrow 0} \frac{-2x-h}{x^2(x+h)^2}$$
$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

$f'(2) = \frac{-2}{2^3} = -\frac{1}{4}$

2 $f'(x) = -\frac{2}{x^3}$

$f'(x) = 2 \Rightarrow \frac{-2}{x^3} = 2$

$\Rightarrow x^3 = -1 \Rightarrow x = -1 \therefore$ point is $(-1, 1)$

3 $f'(x) = 4x - \frac{1}{x^2}$

$f'(x) = 3 \Rightarrow 4x - \frac{1}{x^2} = 3$

$\Rightarrow 4x^3 - 1 = 3x^2 \Rightarrow 4x^3 - 3x^2 - 1 = 0$

$\Rightarrow (x-1)(4x^2+x+1) = 0$

$\Rightarrow x = 1 \therefore$ point is $(1, 3)$

Exercise 4F

- 1 a $y = x^2 + 2x + 1$ $f'(x) = 2x + 2$ $f'(0) = 2$
 b $y = x^3 - 1$ $f'(x) = 3x^2$ $f'(1) = 3$
 c $y = \frac{2}{x}$ $f'(x) = \frac{-2}{x^2}$ $\therefore f'(3) = \frac{-2}{9}$
 d $y = \sqrt{x-1}$ $f'(x) = \frac{1}{2\sqrt{x-1}}$ $\therefore f'(2) = \frac{1}{2}$
 e $y = \sqrt{x+3}$, $f'(x) = \frac{1}{2\sqrt{x+3}}$, $f'(1) = \frac{1}{4}$
 f $y = \frac{1}{\sqrt{x}}$, $f'(x) = -\frac{1}{2\sqrt{x^3}}$, $f'(4) = -\frac{1}{16}$
- 2 a Average velocity = $\frac{x(a+h)-x(a)}{h}$
 $= \frac{12-5(a+h)^2-12+5a^2}{h}$
 $= -10a - 5h$
 b velocity = $\lim_{h \rightarrow 0} (\text{average velocity}) = -10a$

Exercise 4G

- 1 $y = 9 - x^2$ $\frac{dy}{dx} = -2x$
 a When $x = -1$, gradient = 2
 b When $x = -1$, gradient = 8, $\frac{dy}{dx} = 2$
 $\therefore$ tangent is $y - 8 = 2(x + 1)$ i.e. $y = 2x + 10$
 c Normal is $y - 8 = -\frac{1}{2}(x + 1)$ i.e. $y = -\frac{1}{2}x + \frac{15}{2}$
- 2 $y = \frac{1}{x-1}$ $\frac{dy}{dx} = \frac{-1}{(x-1)^2}$ $\therefore \frac{-1}{(x-1)^2} = -1$
 $\therefore (x-1)^2 = 1$ $x-1 = \pm 1$
 $x = 0$ or 2 $(0, -1), (2, 1)$
 At $(0, -1)$ $y + 1 = -1(x - 0)$ $y = -x - 1$
 At $(2, 1)$ $y - 1 = -1(x - 2)$ $y = -x + 3$
- 3 a $y = 4 - 3x - 3x^2$ $\frac{dy}{dx} = -3 - 6x$
 $-3 - 6x = 0$ $\therefore x = x - \frac{1}{2}$ $\left(\frac{-1}{2}, \frac{19}{4}\right)$
 b $y = x^3 + 1$ $\frac{dy}{dx} = 3x^2$
 $3x^2 = 0$ $\therefore x = 0$ $(0, 1)$
 c $y = \frac{1}{x}$ $\frac{dy}{dx} = \frac{-1}{x^2}$ $\frac{-1}{x^2} \neq 0$ $\therefore$ no points
 d $y = x^2 - 3x$ $\frac{dy}{dx} = 2x - 3$
 $2x - 3 = 0$ $\therefore x = \frac{3}{2}$ $\left(\frac{3}{2}, \frac{-9}{4}\right)$
 e $y = \sqrt{x}$ $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ $\frac{1}{2\sqrt{x}} \neq 0$ $\therefore$ no points
- 4 $y = x + \frac{1}{x}$ $\frac{dy}{dx} = 1 - \frac{1}{x^2}$
 At $(1, 2)$ $\frac{dy}{dx} = 1 - 1 = 0$
 Equation of tangent is $y = 2$
 $\therefore$ normal is $x = 1$

Exercise 4H

- 1 a $y = 4 - x - 3x^2$ $\frac{dy}{dx} = -1 - 6x$
 b $y = 2x^4 - 3x + 1$ $\frac{dy}{dx} = 8x^3 - 3$
 c $y = 4x^3 - \frac{1}{x^3} + 2x^2 + \frac{2}{3x^2} = 4x^3 - x^{-3} + 2x^2 + \frac{2}{3}x^{-2}$
 $\frac{dy}{dx} = 12x^2 + 3x^{-4} + 4x - \frac{4}{3}x^{-3}$
 $= 12x^2 + \frac{3}{x^4} + 4x - \frac{4}{3x^3}$
 d $y = \frac{2-3x^2+5x^4}{x} = 2x^{-1} - 3x + 5x^3$
 $\frac{dy}{dx} = -2x^{-2} - 3 + 15x^2 = -\frac{2}{x^2} - 3 + 15x^2$
- 2 $y = 2(3x^2 - 2x) = 6x^2 - 4x$
 $\frac{dy}{dx} = 12x - 4$
 At $(1, -2)$ $\frac{dy}{dx} = 8$
 Equation of tangent: $y - 2 = 8(x - 1)$
 $y = 8x - 6$
- 3 $y = \frac{x-3}{x} = 1 - 3x^{-1}$
 $\frac{dy}{dx} = 3x^{-2} = \frac{3}{x^2}$
 At $(-1, 4)$ $\frac{dy}{dx} = 3$ $\therefore$ gradient of normal = $-\frac{1}{3}$
 Equation of normal: $y - 4 = -\frac{1}{3}(x + 1)$
 $y = -\frac{1}{3}x + \frac{11}{3}$

Exercise 4I

- 1 a $y = (2x + 3)^5$ $\frac{dy}{dx} = 5(2x + 3)^4 (2) = 10(2x + 3)^4$
 b $y = \sqrt{2-3x} = (2-3x)^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}(2-3x)^{-\frac{1}{2}}(-3) = \frac{-3}{2\sqrt{2-3x}}$
 c $y = \frac{2}{x} - 3x + 5x^3$, so $\frac{dy}{dx} = -\frac{2}{x^2} - 3 + 15x^2$
 d $y = \frac{-3}{\sqrt{5x^2+1}} = -3(5x^2+1)^{-\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{3}{2}(5x^2+1)^{-\frac{3}{2}}(10x) = \frac{15x}{\sqrt{(5x^2+1)^3}}$
- 2 $y = \sqrt{3x^2-2x} = (3x^2-2x)^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}(3x^2-2x)^{-\frac{1}{2}}(6x-2) = \frac{3x-1}{\sqrt{3x^2-2x}}$
 At $x = 1$, $\frac{dy}{dx} = 2$ and $y = 1$, so tangent is
 $y - 1 = 2(x - 1)$ i.e. $y = 2x - 1$
- 3 $y = \frac{x-3}{x} = 1 - \frac{3}{x}$
 $\frac{dy}{dx} = \frac{3}{x^2}$
 At $(1, -2)$, $\frac{dy}{dx} = 3$ so gradient of normal = $-\frac{1}{3}$
 $\therefore$ equation of normal is $y + 2 = -\frac{1}{3}(x - 1)$
 i.e. $y = -\frac{1}{3}x - \frac{5}{3}$

$$4 \quad y = \frac{1}{3x^2 - 6x + 1} = (3x^2 - 6x + 1)^{-1}$$

$$\frac{dy}{dx} = -(3x^2 - 6x + 1)^{-2} (6x - 6) = -\frac{(6x - 6)}{(3x^2 - 6x + 1)^2}$$

$$6x - 6 = 0 \quad \therefore x = 1 \quad y = \frac{1}{3 - 6 + 1} = -\frac{1}{2} \quad \left(1, -\frac{1}{2}\right)$$

$$5 \quad y = \sqrt{1 - \sqrt{x}} = \left(1 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(1 - x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(-\frac{1}{2} x^{-\frac{1}{2}}\right) = -\frac{1}{4\sqrt{1 - \sqrt{x}}\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-1}{4\sqrt{\sqrt{x} - x}}$$

Exercise 4J

$$1 \quad y = (x - 1)(x + 3)^3$$

$$u(x) = x - 1 \quad u'(x) = 1$$

$$v(x) = (x + 3)^3 \quad v'(x) = 3(x + 3)^2$$

$$\frac{dy}{dx} = (x - 1)3(x + 3)^2 + (x + 3)^3(1)$$

$$= (x + 3)^2(3(x - 1) + (x + 3))$$

$$= (x + 3)^2(4x)$$

$$= 4x(x + 3)^2$$

$$2 \quad y = (2x - 3)^2(4x + 1)^3$$

$$u(x) = (2x - 3)^2 \quad u'(x) = 4(2x - 3)$$

$$v(x) = (4x + 1)^3 \quad v'(x) = 12(4x + 1)^2$$

$$\frac{dy}{dx} = (2x - 3)^2 12(4x + 1)^2 + (4x + 1)^3 4(2x - 3)$$

$$= 4(2x - 3)(4x + 1)^2 [3(2x - 3) + (4x + 1)]$$

$$= 4(2x - 3)(4x + 1)^2 (10x - 8)$$

$$= 8(2x - 3)(4x + 1)^2 (5x - 4)$$

$$3 \quad y = \frac{x+1}{x-1} = (x+1)(x-1)^{-1}$$

$$u(x) = x + 1 \quad u'(x) = 1$$

$$v(x) = (x - 1)^{-1} \quad v'(x) = -(x - 1)^{-2}$$

$$\frac{dy}{dx} = -(x + 1)(x - 1)^{-2} + (x - 1)^{-1}$$

$$\frac{dy}{dx} = (x - 1)^{-2} [-(x + 1) + (x - 1)] = \frac{-2}{(x - 1)^2}$$

$$4 \quad y = x\sqrt{1 - 2x}$$

$$u(x) = x \quad u'(x) = 1$$

$$v(x) = (1 - 2x)^{\frac{1}{2}} \quad v'(x) = \frac{1}{2}(1 - 2x)^{-\frac{1}{2}}(-2) = -(1 - 2x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -x(1 - 2x)^{-\frac{1}{2}} + (1 - 2x)^{\frac{1}{2}}$$

$$= (1 - 2x)^{-\frac{1}{2}} [-x + (1 - 2x)] = \frac{1 - 3x}{\sqrt{1 - 2x}}$$

$$5 \quad y = (x^4 - 3x + 1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(x^4 - 3x + 1)^2} (4x^3 - 3) = \frac{3 - 4x^3}{(x^4 - 3x + 1)^2}$$

$$6 \quad y = (x - 1)^4(3x - 2)^{\frac{2}{3}}$$

$$\frac{dy}{dx} = (x - 1)^4 \frac{2}{3}(3x - 2)^{-\frac{1}{3}} \times 3 + 4(x - 1)^3(3x - 2)^{\frac{2}{3}}$$

$$= \frac{2(x - 1)^4}{(3x - 2)^{\frac{1}{3}}} + 4(x - 1)^3(3x - 2)^{\frac{2}{3}}$$

$$= \frac{2(x - 1)^3(7x - 5)}{(3x - 2)^{\frac{1}{3}}}$$

Exercise 4K

$$1 \quad a \quad y = \frac{x^2 - 7}{x^3}$$

$$u(x) = x^2 - 7 \quad u'(x) = 2x$$

$$v(x) = x^3 \quad v'(x) = 3x^2$$

$$\frac{dy}{dx} = \frac{x^3(2x) - (x^2 - 7)3x^2}{x^6}$$

$$= \frac{2x^2 - 3x^2 + 21}{x^4}$$

$$= \frac{21 - x^2}{x^4}$$

$$b \quad y = \frac{x}{\sqrt{x^2 + 1}}$$

$$u(x) = x \quad u'(x) = 1$$

$$v(x) = (x^2 + 1)^{\frac{1}{2}} \quad v'(x) = (x^2 + 1)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \left((x^2 + 1)^{\frac{1}{2}} - x^2(x^2 + 1)^{-\frac{1}{2}}\right) \div (x^2 + 1)$$

$$= \frac{(x^2 + 1) - x^2}{(x^2 + 1)^{\frac{3}{2}}} = \frac{1}{(x^2 + 1)^{\frac{3}{2}}}$$

$$c \quad y = \frac{1}{x^4 - 3x + 1} = (x^4 - 3x + 1)^{-1}$$

$$\frac{dy}{dx} = -(x^4 - 3x + 1)^{-2} (4x^3 - 3)$$

$$y = \frac{3 - 4x^3}{(x^4 - 3x + 1)^2}$$

$$d \quad y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$$

$$u(x) = 1 + x^{\frac{1}{2}} \quad u'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$v(x) = 1 - x^{\frac{1}{2}} \quad v'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{(1 - x^{\frac{1}{2}})^{\frac{1}{2}}x^{\frac{1}{2}} + (1 - x^{\frac{1}{2}})^{\frac{1}{2}}x^{-\frac{1}{2}}}{(1 - x^{\frac{1}{2}})^2}$$

$$= \frac{x^{-\frac{1}{2}}}{(1 - x^{\frac{1}{2}})^2} = \frac{1}{\sqrt{x}(1 - \sqrt{x})^2}$$

$$e \quad y = (\sqrt{x} - x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(\sqrt{x} - x)^{-\frac{1}{2}} \left(\frac{1}{2}x^{-\frac{1}{2}} - 1\right)$$

$$= \frac{1}{4}(\sqrt{x} - x)^{\frac{1}{2}} \left(x^{-\frac{1}{2}} - 2 \right)$$

$$= \frac{x^{-\frac{1}{2}} - 2}{4(\sqrt{x} - x)^{\frac{1}{2}}} = \frac{1 - 2\sqrt{x}}{4\sqrt{x}(\sqrt{x} - x)} = \frac{1 - 2\sqrt{x}}{4\sqrt{x} - x\sqrt{x}}$$

f $y = \left(\frac{x}{1 - \sqrt{x}} \right)^3 = \frac{x^3}{(1 - \sqrt{x})^3}$

$$u(x) = x^3 \quad u'(x) = 3x^2$$

$$v(x) = (1 - \sqrt{x})^3 \quad v'(x) = 3(1 - \sqrt{x})^2 \left(-\frac{1}{2} x^{-\frac{1}{2}} \right)$$

$$\frac{dy}{dx} = \frac{3x^2(1 - \sqrt{x})^3 + \frac{3}{2}x^{\frac{5}{2}}(1 - \sqrt{x})^2}{(1 - \sqrt{x})^6}$$

$$= \frac{3x^2(1 - \sqrt{x}) + \frac{3}{2}x^{\frac{5}{2}}}{(1 - \sqrt{x})^4}$$

$$= \frac{6x^2 - 6x^{\frac{5}{2}} + 3x^{\frac{5}{2}}}{2(1 - \sqrt{x})^4}$$

$$= \frac{6x^2 + 3x^{\frac{5}{2}}}{2(1 - \sqrt{x})^4}$$

$$= \frac{3x^2(2 - \sqrt{x})}{2(1 - \sqrt{x})^4}$$

2 $y = \frac{4x}{x^2 + 1} \quad (x = -1)$

$$u(x) = 4x \quad u'(x) = 4$$

$$v(x) = x^2 + 1 \quad v'(x) = 2x$$

$$\frac{dy}{dx} = \frac{4(x^2 + 1) - 8x^2}{(x^2 + 1)^2} = \frac{4 - 4x^2}{(x^2 + 1)^2}$$

At $x = -1$, gradient = 0

3 $y = \frac{8}{4 + x^2} = 8(4 + x^2)^{-1} \quad (x = 1)$

$$\frac{dy}{dx} = -8(4 + x^2)^{-2} (2x) = \frac{-16x}{(4 + x^2)^2}$$

At $\left(1, \frac{8}{5}\right)$, $\frac{dy}{dx} = \frac{-16}{25}$ gradient of normal = $\frac{25}{16}$

Equation of normal: $y - \frac{8}{5} = \frac{25}{16}(x - 1)$, $y = \frac{25}{16}x + \frac{3}{80}$

4 $f(x) = \sqrt[3]{\left(1 - \frac{1}{2 + x}\right)^2} = \left(1 - (2 + x)^{-1}\right)^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3} \left(1 - (2 + x)^{-1}\right)^{-\frac{1}{3}} (2 + x)^{-2}$$

$$= \frac{2}{3(2 + x)^2 \sqrt[3]{1 - \frac{1}{2 + x}}}$$

Exercise 4L

1 $f(x) = 4x + 1 + \frac{1}{x}$

$$f'(x) = 4 - x^{-2} \quad f''(x) = 2x^{-3} = \frac{2}{x^3}$$

2 $f(x) = x^4 - 2x - 1$

$$f'(x) = 4x^3 - 2 \quad f''(x) = 12x^2$$

$$f'(0) = -2 \quad f''(-1) = 12$$

3 $f(x) = x^4 - 4x^3 + 16x - 16$

$$f'(x) = 4x^3 - 12x^2 + 16$$

$$f''(x) = 12x^2 - 24x$$

$$f(x) = (x + 2)(x - 2)^3 \Rightarrow x = -2 \text{ or } x = 2$$

$$f(-2) \neq f'(-2) \neq f''(-2),$$

but $f(2) = f'(2) = f''(2) = 0$ so $x = 2$

4 $f(x) = x^4 + rx^2 + sx + t$

$$f(-1) = 16 \Rightarrow r - s + t = 15 \quad (1)$$

$$f'(x) = 4x^3 + 2rx + s$$

$$f'(-1) = -16 \Rightarrow s - 2r = -12 \quad (2)$$

$$f''(x) = 12x^2 + 2r$$

$$f''(-1) = 16 \Rightarrow 12 + 2r = 16 \quad (3)$$

Solve equations (1), (2), (3) to find $r = 2$, $s = -8$, $t = 5$.

5 $s(t) = (t - 4)^3 (3 - 2t)^2$

a Velocity = $s'(t) = (t - 4)^3 2(3 - 2t) (-2)$

$$= 3(t - 4)^2 (3 - 2t)^2$$

$$s'(t) = (t - 4)^2 (3 - 2t)[-4(t - 4) + 3(3 - 2t)]$$

$$= (t - 4)^2 (3 - 2t) (25 - 10t)$$

$$s'(4) = 0 \text{ ms}^{-1}$$

b $s'(t) = (t - 4)^2 (75 - 80t + 20t^2)$

$$s''(t) = (t - 4)^2 (-80 + 40t) + 2(t - 4) (70 - 80t + 20t^2)$$

$$= (t - 4) [-80t + 40t^2 + 320 - 160t + 150 - 160t + 40t^2]$$

$$= (t - 4) (80t^2 - 400t + 470)$$

$$\text{acceleration} = s''(4) = 0 \text{ ms}^{-2}$$

c $s''(t) = 80t^3 - 720t^2 + 2070t - 1880$

$$\text{jerk} = s'''(t) = 240t^2 - 1440t + 2070$$

$$s'''(1) = 240 - 1440 + 2070 = 870 \text{ ms}^{-1}$$

6 $f(x) = \frac{1}{x} = x^{-1}$

$$f'(x) = -x^{-2} \quad f''(x) = 2x^{-3} \quad f'''(x) = -6x^{-4}$$

$$f^{(4)}(x) = 24x^{-5} \quad f^{(5)}(x) = -120x^{-6}$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

Exercise 4M

1 a $y = x^2 - 3x + 1$

$$\frac{dy}{dx} = 2x - 3$$

$$2x - 3 = 0 \quad \therefore x = \frac{3}{2} \quad y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1 = -\frac{5}{4}$$

x	$x < \frac{3}{2}$	$x > \frac{3}{2}$
$\frac{dy}{dx}$	-	+

$$\therefore \text{minimum value} = -\frac{5}{4} \quad \left(\text{when } x = \frac{3}{2}\right)$$

b $y = -2x^3 + 6x^2 - 3$

$$\frac{dy}{dx} = -6x^2 + 12x$$

$$-6x^2 + 12x = 0 \quad \therefore 6x(-x + 2) = 0$$

$$x = 0 \quad \text{or } 2 \quad (0, -3) \quad (2, 5)$$

x	$x < 0$	$0 < x < 2$	$x > 2$
$\frac{dy}{dx}$	-	+	-

$$\therefore \text{minimum value} = -3 \quad (\text{at } x = 0)$$

$$\text{maximum value} = 5 \quad (\text{at } x = 2)$$

c $y = 3x^4 - 2x^3 - 3x^2 + 4$

$$\frac{dy}{dx} = 12x^3 - 6x^2 - 6x$$

$$12x^3 - 6x^2 - 6x = 0 \quad 6x(2x^2 - x - 1) = 0$$

$$6x(2x + 1)(x - 1) = 0$$

$$x = 0, \quad -\frac{1}{2} \text{ or } 1 \quad (0, 4) \quad \left(-\frac{1}{2}, \frac{59}{16}\right) \quad (1, 2)$$

x	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 0$	$0 < x < 1$	$x > 1$
$\frac{dy}{dx}$	-	+	-	+

$$\therefore \text{minimum values are } \frac{59}{16} \text{ at } x = -\frac{1}{2} \text{ and}$$

$$2 \text{ (at } x = 1) \text{ maximum value} = 4 \text{ (at } x = 0)$$

d $y = x^4 - 4x^3$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$4x^3 - 12x^2 = 0 \quad 4x^2(x - 3) = 0$$

$$x = 0 \text{ or } 3 \quad (0, 0) \quad (3, -27)$$

x	$x < 0$	$0 < x < 3$	$x > 3$
$\frac{dy}{dx}$	-	-	+

$$\therefore \text{horizontal point of inflexion at } (0, 0)$$

$$\text{minimum value} = -27 \text{ (at } x = 3)$$

Exercise 4N

1 a i $x = -1$ or 1 ii $]-\infty, -1[\cup]1, \infty[$

iii $]-1, 1[$

b i $x = -1, \frac{1}{2}, \frac{3}{2}$ ii $]-1, \frac{1}{2}[\cup]\frac{3}{2}, \infty[$

iii $]-\infty, -1[\cup]\frac{1}{2}, \frac{3}{2}[$

2 a i $x = -\frac{1}{2}$ ii $]-\frac{1}{2}, \infty[$

iii $]-\infty, -\frac{1}{2}[$

3 a $y = -3x^2 + 6x - 1$

$$\frac{dy}{dx} = -6x + 6 \quad -6x + 6 = 0 \Rightarrow x = 1 \quad (1, 2)$$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	-

maximum at $(1, 2)$

f is increasing for $-\infty, 1[$

f is decreasing for $1, \infty[$

b $y = x\sqrt{2-x^2} \quad (-\sqrt{2} \leq x \leq \sqrt{2})$

$$\frac{dy}{dx} = x \cdot \frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x) + (2-x^2)^{\frac{1}{2}}$$

$$= \frac{-x^2}{(2-x^2)^{\frac{1}{2}}} + (2-x^2)^{\frac{1}{2}}$$

$$= \frac{-x^2 + (2-x^2)}{\sqrt{2-x^2}} = \frac{2-2x^2}{\sqrt{2-x^2}}$$

$$\frac{2-2x^2}{\sqrt{2-x^2}} = 0 \Rightarrow x = \pm 1 \quad (1, 1) \quad (-1, -1)$$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	-	0	+	0	-

minimum at $(-1, -1)$, maximum at $(1, 1)$

increasing for $]-1, 1[$

decreasing for $[-\sqrt{2}, -1[\cup]1, \sqrt{2}]$

c $y = \frac{x}{x^2+1} \quad \frac{(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$$\frac{dy}{dx} = 0 \quad \text{when } x = \pm 1 \quad \left(1, \frac{1}{2}\right) \quad \left(-1, -\frac{1}{2}\right)$$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	-	0	+	0	-

minimum at $(-1, -\frac{1}{2})$, maximum at $(1, \frac{1}{2})$

increasing for $]-1, 1[$

decreasing for $]-\infty, -1[\cup]1, \infty[$

d $y = x^{\frac{1}{3}}(x-2) = x^{\frac{4}{3}} - 2x^{\frac{1}{3}}$
 $\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}}, \quad \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}} = 0$
 $\therefore 4x - 2 = 0 \quad \therefore x = \frac{1}{2} \left(\frac{1}{2}, \frac{-3}{2(2)^{\frac{1}{3}}} \right)$

x	$x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	-	0	+

minimum at $\left(\frac{1}{2}, \frac{3}{2^{\frac{4}{3}}} \right)$

increasing for $]\frac{1}{2}, \infty[$, decreasing for $]-\infty, \frac{1}{2}[$

e $y = x^2\sqrt{2-x^2} \quad (-\sqrt{2} \leq x \leq \sqrt{2})$
 $\frac{dy}{dx} = x^2 \cdot \frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x) + 2x(2-x^2)^{\frac{1}{2}}$
 $= \frac{-x^3 + 2x(2-x^2)}{(2-x^2)^{\frac{1}{2}}} = \frac{4x-3x^3}{\sqrt{2-x^2}}$

$\frac{dy}{dx} = 0 \Rightarrow 4x - 3x^3 = 0, x(4-3x^2) = 0$

$x = 0, \pm \frac{2}{\sqrt{3}} \quad (0, 0), \left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}} \right), \left(-\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}} \right)$

x	$x < -\frac{2}{\sqrt{3}}$	$x = -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < 0$	$x = 0$
$\frac{dy}{dx}$	+	0	-	0

x	$0 < x < \frac{2}{\sqrt{3}}$	$x = \frac{2}{\sqrt{3}}$	$x > \frac{2}{\sqrt{3}}$
$\frac{dy}{dx}$	+	0	-

maxima at $\left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}} \right)$ and $\left(-\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}} \right)$

minimum at $(0, 0)$,

increasing for $]-\sqrt{2}, -\frac{2}{\sqrt{3}}[\cup]0, \frac{2}{\sqrt{3}}[$

decreasing for $]-\frac{2}{\sqrt{3}}, 0[\cup]\frac{2}{\sqrt{3}}, \sqrt{2}]$

Exercise 40

1 a $y = 2x^3 + 3x^2 - 12x - 3$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$6(x^2 + x - 2) = 0$

$6(x+2)(x-1) = 0$

$x = -2$ or 1

$\frac{d^2y}{dx^2} = 12x + 6$

$f''(-2) = -18 < 0 \quad \therefore f$ has a maximum at $(-2, 17)$

$f''(1) = 18 > 0 \quad \therefore f$ has a minimum at $(1, -10)$

b $y = -x^4 + 2x - 1$

$\frac{dy}{dx} = -4x^3 + 2 \quad -4x^3 + 2 = 0 \quad \therefore x^3 = \frac{1}{2} \quad x = \frac{1}{\sqrt[3]{2}}$
 $= 0.794$

$\frac{d^2y}{dx^2} = -12x^2 < 0 \quad \therefore$ maximum at $(0.794, 0.191)$

c $y = x^5 - 5x$

$\frac{dy}{dx} = 5x^4 - 5 \quad 5x^4 - 5 = 0, \quad x^4 = 1, \quad x = \pm 1$

$\frac{d^2y}{dx^2} = 20x^3$

$f''(-1) < 0 \quad \therefore$ maximum at $(-1, 4)$

$f''(1) > 0 \quad \therefore$ minimum at $(1, -4)$

d $y = \frac{12}{x^2 + 2x - 3} = 12(x^2 + 2x - 3)^{-1}$

$\frac{dy}{dx} = -12(x^2 + 2x - 3)^{-2}(2x + 2) = \frac{-24(x+1)}{(x^2 + 2x - 3)^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = -1$

x	$x < -1$	$x = -1$	$x > -1$
$\frac{dy}{dx}$	+	0	-

maximum at $(-1, -3)$

e $y = \frac{3x+3}{x(3-x)} = \frac{3x+3}{3x-x^2}$

$\frac{dy}{dx} = \frac{(3x-x^2)3 - (3x+3)(3-2x)}{(3x-x^2)^2}$

$= \frac{9x-3x^2-9x+6x^2-9+6x}{(3x-x^2)^2}$

$= \frac{3x^2+6x-9}{x^2(3-x)^2} = \frac{3(x^2+2x-3)}{x^2(3-x)^2} = \frac{3(x+3)(x-1)}{x^2(3-x)^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = -3$ or 1

x	$x < -3$	$x = -3$	$-3 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	-	0	+

maximum at $\left(-3, \frac{1}{3} \right)$, minimum at $(1, 3)$

2 b $y = \frac{1-x}{x^2+8}$

i $\frac{dy}{dx} = \frac{(-x^2+8)-2x(1-x)}{(x^2+8)^2} = \frac{x^2-2x-8}{(x^2+8)^2}$

$\frac{dy}{dx} = 0 \Rightarrow (x-4)(x+2) = 0$
 $x = -2$ or 4

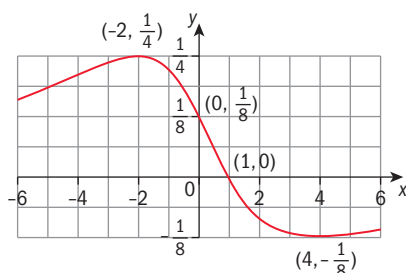
x	$x < -2$	$x = -2$	$-2 < x < 4$	$x = 4$	$x > 4$
$\frac{dy}{dx}$	+	0	-	0	+

maximum at $\left(-2, \frac{1}{4} \right)$ minimum at $\left(4, \frac{-1}{8} \right)$

ii f is increasing for $]-\infty, -2[\cup]4, \infty[$

f is decreasing for $]-2, 4[$

iii



Exercise 4P

1 a $y = x^3 - x$

i $\frac{dy}{dx} = 3x^2 - 1$ $\frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	-	0	+

point of inflexion at $(0, 0)$

ii concave up for $]0, \infty[$

concave down for $]-\infty, 0[$

b $y = x^4 - 3x + 2$

i $\frac{dy}{dx} = 4x^3 - 3$ $\frac{d^2y}{dx^2} = 12x^2$

$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	+	0	+

no point of inflexion

ii concave up for $]-\infty, 0[\cup]0, \infty[$

c $y = \sqrt{4x - x^2}$ where $x \in [0, 4]$

i $\frac{dy}{dx} = \frac{1}{2}(4x - x^2)^{-\frac{1}{2}}(4 - 2x) = \frac{2 - x}{(4x - x^2)^{\frac{1}{2}}}$

$\frac{d^2y}{dx^2} = \frac{-(4x - x^2)^{\frac{1}{2}} - (2 - x)\frac{1}{2}(4x - x^2)^{-\frac{1}{2}}(4 - 2x)}{(4x - x^2)}$

$= \frac{-(4x - x^2) - (2 - x)^2}{(4x - x^2)^{\frac{3}{2}}}$

$= \frac{-4x + x^2 - 4 + 4x - x^2}{(4x - x^2)^{\frac{3}{2}}}$

$= \frac{-4}{(4x - x^2)^{\frac{3}{2}}}$

$\frac{d^2y}{dx^2} < 0 \therefore$ no points of inflexion

ii concave down for $[0, 4]$

d $y = (x - 1)^{\frac{2}{3}}$

i $\frac{dy}{dx} = \frac{2}{3}(x - 1)^{-\frac{1}{3}}$ $\frac{d^2y}{dx^2} = -\frac{2}{9}(x - 1)^{-\frac{4}{3}}$

$\frac{d^2y}{dx^2} = \frac{-2}{9(x - 1)^{\frac{4}{3}}} \neq 0$

$\therefore$ no points of inflexion

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	-	0

concave down for $]-\infty, 1[\cup]1, \infty[$

e $y = \frac{3x^2}{x - 1}$

i $\frac{dy}{dx} = \frac{6x(x - 1) - 3x^2}{(x - 1)^2} = \frac{3x^2 - 6x}{(x - 1)^2}$

$\frac{d^2y}{dx^2} = \frac{(x - 1)^2(6x - 6) - (3x^2 - 6x)2(x - 1)}{(x - 1)^4}$

$= \frac{(x - 1)(6x - 6) - (3x^2 - 6x)2}{(x - 1)^3}$

$= \frac{6x^2 - 12x + 6 - 6x^2 + 12x}{(x - 1)^3}$

$= \frac{6}{(x - 1)^3}$

$\frac{d^2y}{dx^2} \neq 0 \therefore$ no points of inflexion

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	-	+

concave up for $]1, \infty[$, concave down for $]-\infty, 1[$

Exercise 4Q

1 $s(t) = -5t^2 + 5t + 10$

a $s(0) = 10$ m

b $-5t^2 + 5t + 10 = 0$

$t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

$\therefore t = 2$ seconds

c $v(t) = -10t + 5$ $a(t) = -10$

$v(2) = -15 \text{ ms}^{-1}$ $a(2) = -10 \text{ ms}^{-2}$

The diver is moving downwards and speeding up when he hits the water.

2 $s = 50t - 15t^2$

a $v = 50 - 30t = 0$ when $t = \frac{5}{3}$

maximum height $= 5\left(\frac{5}{3}\right) = 41\frac{2}{3}$ m

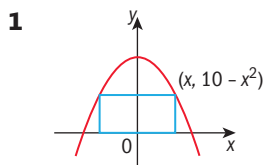
- b** $20 = 50t - 15t^2$
 $3t^2 - 10t + 4 = 0$
 $t = 0.4648$ or 2.8685
 $v = 50 - 30t$ $v(0.4648) = 36.1 \text{ ms}^{-1}$
 $v(2.8685) = -36.1 \text{ ms}^{-1}$
 speed = 36.1 ms^{-1} upwards (when $t = 0.4648$)
 and downwards (when $t = 2.8685$)
- c** $a = -30 \text{ ms}^{-2}$
- d** $50t - 15t^2 = 0$
 $5t(10 - 3t) = 0$
 $\therefore$ rock hits the ground again when $t = \frac{10}{3} \text{ s}$
- 3** $s = 7t + 5t^2 - 2t^3$
a $v = 7 + 10t - 6t^2$ $a = 10 - 12t$
 $v(0) = 7 \text{ ms}^{-1}$ $a(0) = -10 \text{ ms}^{-2}$
 Initially the particle is moving in a positive direction and is slowing down.
- b** $v(2) = 3 \text{ ms}^{-1}$ $a(2) = -14 \text{ ms}^{-2}$
 The particle is moving in a positive direction and slowing down.
- 4** $s = 10t^2 - t^3$
a $s(3) = 63 \text{ m}$ $\therefore$ average velocity = $\frac{63}{3} = 21 \text{ ms}^{-1}$
- b** $v = 20t - 3t^2$ $a = 20 - 6t$
 $v(3) = 33 \text{ ms}^{-1}$ $a(3) = 2 \text{ ms}^{-2}$
- c** Speeding up.
- d** $20t - 3t^2 = 0$
 $t(20 - 3t) = 0$
 $t = 0$ or $\frac{20}{3}$ direction changes when $t = \frac{20}{3} \text{ s}$
- 5** $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t$
a $v(t) = t^2 - 6t + 8$ $a(t) = 2t - 6$
- b** **i** $t^2 - 6t + 8 = 0$
 $(t - 2)(t - 4) = 0$
 $t = 2 \text{ s}$ or 4 s
- ii**
- | t | $0 < t < 2$ | $2 < t < 4$ | $t > 4$ |
|-----|-------------|-------------|---------|
| v | + | - | + |
| t | $0 < t < 3$ | $t > 3$ | |
| a | - | + | |
- v and a have the same sign for $2 < t < 3$ and $t > 4$ $\therefore$ the particle is speeding up at these times.
- iii** the particle is slowing down for $0 < t < 2$ and $3 < t < 4$
- c** $a(2) = -2 \text{ ms}^{-1}$ $a(4) = 2 \text{ ms}^{-2}$
 the particle changes direction from positive to negative when $t = 2 \text{ s}$ and from negative to positive when $t = 4 \text{ s}$.

- d** $t = 2 \text{ s}$ and $t = 4 \text{ s}$.
- e** $0 - 2 \text{ s}$ distance = $s(2) - s(0) = 6\frac{2}{3} - 0 = 6\frac{2}{3}$
 $2 - 4 \text{ s}$ distance = $s(4) - s(2) = 5\frac{1}{3} - 6\frac{2}{3} = -1\frac{1}{3}$
 $4 - 5 \text{ s}$ distance = $s(5) - s(4) = 6\frac{2}{3} - 5\frac{1}{3} = 1\frac{1}{3}$
 total distance = $6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 9\frac{1}{3} \text{ m}$

Exercise 4R

- 1** $c(x) = 20000 + 180x - 0.1x^2$
a $c'(x) = 180 - 0.2x$
b $c'(100) = 180 - 0.2 \times 100$
 $= 160 \text{ euros / tank}$
c $c(101) - c(100) = 159.9 \Rightarrow$ cost of producing 1 extra tank is nearly the same as the marginal cost function.
- 2** **a** **i** $p(x)$ must be > 0 so $0.002x < 7$ i.e. $x < 3500$
 $\therefore$ domain is $0 < x < 3500$
ii $c'(x) = 3 \text{ euros / unit} \Rightarrow$ it will always cost 3 euros to make an extra memory stick
iii $r(x) = x(7 - 0.002x)$
b Break-even points: $r(x) = c(x)$ when
 $7x = 0.002x^2 = 500 + 3x$
 i.e. $0.002x^2 - 4x + 500 = 0$
 $\Rightarrow x = \frac{4 \pm \sqrt{16 - 4}}{0.004}$
 $= 134$ or 1870 (3 sf).
 For profit, need to make x memory sticks where $134 < x < 1870$.
- 3** Average cost per 100 units = $\frac{c(x)}{x} = 500 + \frac{1000}{x}$
 To minimize cost, need $\frac{d}{dx} \left(\frac{c(x)}{x} \right) = 0$
 $\Rightarrow 500 - \frac{1000}{x^2} = 0$
 $\Rightarrow x^2 = 2 \Rightarrow x = 1.41$ (3 sf)
 $\therefore$ costs are minimised by making 141 units.
- 4** **b** $r(x) = 35x - 3$ and $p(x) = r(x) - c(x)$
 so $p(x) = 35x - 3 - 400 - 20x + 0.2x^2 - 0.0004x^3$
 $= 15x - 403 + 0.2x^2 - 0.0004x^3$
 $p'(x) = 15 + 0.4x - 0.0012x^2 = 0$
 $\Rightarrow x = 367$, so 367 jackets must be made to maximise profit.
- c** Minimising costs will not necessarily maximise profits.

Exercise 4S



$$A = 2x(10 - x^2)$$

$$= 20x - 2x^3$$

$$A' = 20 - 6x^2$$

$$20 - 6x^2 = 0 \Rightarrow x = \sqrt{\frac{10}{3}}$$

$$A'' = -12x$$

$$\text{If } x = \sqrt{\frac{10}{3}}, \quad A'' < 0 \quad \therefore \text{max.}$$

$$\text{base} = 2\sqrt{\frac{10}{3}} \quad \text{height} = 10 - \frac{10}{3} = \frac{20}{3} \quad (3.65 \text{ by } 6.67)$$



$$A = x(800 - 2x)$$

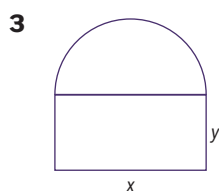
$$= 800x - 2x^2$$

$$A' = 800 - 4x$$

$$A' = 0 \Rightarrow x = 200$$

$$A'' = -4 < 0 \quad \therefore \text{maximum}$$

$$\text{maximum area} = 200 \times 400 = 80\,000 \text{ m}^2$$



$$x + 2y + \frac{\pi x}{2} = 4$$

$$2x + 4y + \pi x = 8$$

$$y = \frac{8 - 2x - \pi x}{4}$$

$$A = xy = \frac{1}{4}(8x - 2x^2 - \pi x^2)$$

$$A' = \frac{1}{4}(8 - 4x - 2\pi x)$$

$$A = 0 \Rightarrow x(4 + 2\pi) = 8$$

$$x = \frac{8}{4 + 2\pi} = 0.778 \quad y = 1$$

$$A'' = \frac{1}{4}(-4 - 2\pi) < 0 \quad \therefore \text{maximum}$$

$$\text{dimensions: } \frac{8}{4 + 2\pi} \text{ m by } 1 \text{ m}$$

Exercise 4T

1 a $3y^2 + x^2 = 4$

$$6y \frac{dy}{dx} + 2x = 0 \quad \therefore \frac{dy}{dx} = \frac{-x}{3y}$$

b $y^4 = x^3 + 1$

$$4y^3 \frac{dy}{dx} = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{4y^3}$$

c $x^2 + y^2 - 3x + 4y = 2$

$$2x + 2y \frac{dy}{dx} - 3 + 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3 - 2x}{2y + 4}$$

d $2x^2 - 3x^2y^2 + y^2 = 9$

$$4x - 3(x^2 \cdot 2y \frac{dy}{dx} + 2xy^2) + 2y \frac{dy}{dx} = 0$$

$$4x - 6x^2y \frac{dy}{dx} - 6xy^2 + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3xy^2 - 2x}{y - 3x^2y}$$

e $(x + y)^2 = 5 - 2x$

$$2(x + y) \left(1 + \frac{dy}{dx}\right) = -2$$

$$1 + \frac{dy}{dx} = -\frac{1}{x + y}$$

$$\frac{dy}{dx} = -\frac{1}{x + y} - 1 = -\frac{(1 + x + y)}{x + y}$$

f $x^2 = \frac{x - y}{x + y} \quad x^3 + x^2y = x - y$

$$3x^2 + x^2 \frac{dy}{dx} + 2xy = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}(x^2 + 1) = 1 - 3x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$$

2 $x^2 - y^2 = 9 \quad (5, 4)$

$$2x - 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{x}{y} = \frac{5}{4}$$

$$y - 4 = \frac{5}{4}(x - 5)$$

$$y = \frac{5}{4}x - \frac{9}{4}$$

3 $y^2 = 3x + 1 \quad (1, -2)$

$$2y \frac{dy}{dx} = 3 \quad \frac{dy}{dx} = \frac{3}{2y} = \frac{-3}{4}$$

$$y + 2 = \frac{4}{3}(x - 1)$$

$$y = \frac{4}{3}x - \frac{10}{3}$$

4 $x^2 - \sqrt{3}xy + 2y^2 = 5 \quad (\sqrt{3}, 2)$

$$2x - \sqrt{3} \left(x \frac{dy}{dx} + y \right) + 4y \frac{dy}{dx} = 0$$

$$-\sqrt{3} \frac{dy}{dx} + 4y \frac{dy}{dx} = \sqrt{3}y - 2x$$

$$\frac{dy}{dx} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x} = 0$$

$$\text{Tangent: } y = 2 \quad \text{Normal: } x = \sqrt{3}$$

5 $x^2 + y^2 - 6x - 8y = 0$

$$2x + 2y \frac{dy}{dx} - 6 - 8 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3 - x}{y - 4} \quad \frac{dy}{dx} = 0 \Rightarrow x = 3$$

$$9 + y^2 - 18 - 8y = 0$$

$$y^2 - 8y - 9 = 0$$

$$(y - 9)(y + 1) = 0$$

$$y = 9 \text{ or } -1$$

stationary points (3, 9), (3, -1)

6 $3x^2 + 2xy + y^2 = 3 \quad (1, -2)$

$$6x + 2x \frac{dy}{dx} + 2y + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(3x + y)}{x + y} = -\frac{(3 - 2)}{-1} = 1$$

$$3x + x \frac{dy}{dx} + y + y \frac{dy}{dx} = 0$$

$$3 + x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$3 + \frac{d^2y}{dx^2} + 2 - 2 \frac{d^2y}{dx^2} + 1 = 0$$

$$\frac{d^2y}{dx^2} = 6$$

7 $x^2 + xy + y^2 = 3$

$$y = 0 \quad x^2 = 3 \quad x = \pm\sqrt{3}$$

$$2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}$$

$$\text{At } (\sqrt{3}, 0) \frac{dy}{dx} = -2 \quad \text{At } (-\sqrt{3}, 0) \frac{dy}{dx} = -2$$

$\therefore$ tangents are parallel

9 $x + y = x^2 - 2xy + y^2$

a $1 + \frac{dy}{dx} = 2x - 2\left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx}$

$$1 + \frac{dy}{dx} = 2x - 2x \frac{dy}{dx} - 2y + 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} + 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 2x - 2y - 1$$

$$\frac{dy}{dx} = \frac{2x - 2y - 1}{2x - 2y + 1}$$

b $1 - \frac{dy}{dx} = 1 - \frac{(2x - 2y - 1)}{(2x - 2y + 1)}$

$$= \frac{2x - 2y + 1 - 2x + 2y + 1}{2x - 2y + 1}$$

$$1 - \frac{dy}{dx} = \frac{2}{2x - 2y + 1}$$

c $\frac{d^2y}{dx^2} = \frac{(2x - 2y + 1)\left(2 - 2 \frac{dy}{dx}\right) - (2x - 2y - 1)\left(2 - 2 \frac{dy}{dx}\right)}{(2x - 2y + 1)^2}$

$$= \frac{4 - 4 \frac{dy}{dx}}{(2x - 2y + 1)^2}$$

$$= \frac{4 - 4 \frac{(2x - 2y - 1)}{(2x - 2y + 1)}}{(2x - 2y + 1)^2}$$

$$= \frac{4(2x - 2y + 1) - 4(2x - 2y - 1)}{(2x - 2y + 1)^3}$$

$$= \frac{8}{(2x - 2y + 1)^3} = \left(\frac{2}{2x - 2y + 1}\right)^3$$

$$\therefore \frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3 \quad (\text{from b})$$

Exercise 4U

1 $A = \pi r^2 \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2 $A = 2\pi r^2 + 2\pi rh$

$$\frac{dA}{dt} = 4\pi r \frac{dr}{dt} + 2\pi r \frac{dh}{dt} + 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt}$$

3 Let x = diagonal of the box.

$$x^2 = l^2 + w^2 + h^2$$

$$2x \frac{dx}{dt} = 2l \frac{dl}{dt} + 2w \frac{dw}{dt} + 2h \frac{dh}{dt}$$

$$\frac{dx}{dt} = \frac{l \frac{dl}{dt} + w \frac{dw}{dt} + h \frac{dh}{dt}}{(l^2 + w^2 + h^2)^{\frac{1}{2}}}$$

4 $\frac{dl}{dt} = 2 \text{ cm s}^{-1} \quad \frac{dw}{dt} = -2 \text{ cm s}^{-1} \quad l = 12 \text{ cm}, w = 5 \text{ cm}$

a $A = lw$

$$\begin{aligned} \frac{dA}{dt} &= l \frac{dw}{dt} + w \frac{dl}{dt} \\ &= 12(-2) + 5(2) \\ &= -14 \text{ cm}^2 \text{ s}^{-1} \end{aligned}$$

b $p = 2l + 2w$

$$\begin{aligned} \frac{dp}{dt} &= 2 \frac{dl}{dt} + 2 \frac{dw}{dt} \\ &= 2(2) + 2(-2) \\ &= 0 \text{ cm s}^{-1} \end{aligned}$$

c Let x = diagonal of the rectangle.

$$x^2 = l^2 + w^2$$

$$2x \frac{dx}{dt} = 2l \frac{dl}{dt} + 2w \frac{dw}{dt} \quad (l = 12, w = 5, x = 13)$$

$$13 \frac{dx}{dt} = 12(2) + 5(-2)$$

$$\frac{dx}{dt} = \frac{14}{13} \text{ cm s}^{-1}$$

5 $\frac{dv}{dt} = 1.5 \text{ m}^3 \text{ s}^{-1} \quad v = 81 \text{ m}^3 \quad x = \sqrt[3]{81} \text{ m}$

Let x = side length of cube

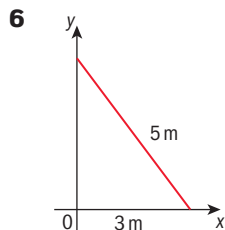
$$V = x^3 \quad A = 6x^2$$

$$\frac{dA}{dt} = \frac{dv}{dt} \times \frac{dx}{dv} \times \frac{dA}{dx}$$

$$= 1.5 \times \frac{1}{3x^2} \times 12x$$

$$= \frac{6}{x}$$

$$\frac{dA}{dt} = \frac{6}{\sqrt[3]{81}} = 1.39 \text{ m}^2 \text{ s}^{-1}$$



When $x = 3 \text{ m}$, $\frac{dx}{dt} = 0.5 \text{ ms}^{-1}$

a $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$3(0.5) + 4 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -0.375$$

$\therefore 0.375 \text{ ms}^{-1}$ down the wall

b $A = \frac{1}{2}xy \quad \frac{dA}{dt} = \frac{1}{2}x \frac{dy}{dt} + \frac{1}{2}y \frac{dx}{dt}$

$$= \frac{1}{2}(3)(-0.375) + \frac{1}{2}(4)(0.5)$$

$$= 0.4375 \text{ m}^2\text{s}^{-1}$$

7 $\frac{dA}{dt} = 2 \text{ cm}^2\text{s}^{-1}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad (r = 5)$$

$$2 = 10\pi \frac{dr}{dt} \quad \frac{dr}{dt} = \frac{1}{5\pi} = 0.0637 \text{ cms}^{-1}$$

10 $\frac{dr_1}{dt} = 1.2 \text{ ms}^{-1} \quad \frac{dr_2}{dt} = 1.5 \text{ ms}^{-1}$

$$A = \pi r_1^2 - \pi r_2^2$$

$$\frac{dA}{dt} = 2\pi r_1 \frac{dr_1}{dt} - 2\pi r_2 \frac{dr_2}{dt}$$

$$= 2\pi(9 \times 1.2 - 1 \times 1.5)$$

$$= 18.6\pi = 58.4 \text{ m}^2\text{s}^{-1}$$



Review exercise

1 a $\lim_{x \rightarrow 1} \frac{x^3 - 3}{x + 1} = -1$

b $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 - 1}}{x}$ does not exist since the domain is $]-\infty, -1] \cup [1, \infty[$

c $\lim_{x \rightarrow 0} \frac{3^x - 1}{x} = 1.10 \text{ (3sf)}$

d $\lim_{x \rightarrow 0} \frac{3x^2 + x^2}{x^2} = 4$

e $\lim_{x \rightarrow \infty} \frac{5x^2}{2x^3 + 1} = 0$

f $\lim_{x \rightarrow -\infty} \frac{7}{x^3 + 1} = 0$

2 $y = \begin{cases} x^2 + 2x & x \leq 2 \\ x^3 - 6x & x > 2 \end{cases}$

$$f(2) = 8 \quad \lim_{x \rightarrow 2^-} f(x) = -4$$

$f(x)$ is not continuous at $x = 2$

3 $a_n = \frac{2n^2 - 3}{n^3 - 2}$ sequence converges since $a_n \rightarrow 0$ as $n \rightarrow \infty$.

4 $\sum_{n=0}^{\infty} 3 \left(\frac{(-1)^n}{5^n} \right)$ is a geometric series with $r = -\frac{1}{5}$ hence it converges.

$$\sum_{n=0}^{\infty} 3 \left(\frac{(-1)^n}{5^n} \right) = \frac{3}{1 - (-\frac{1}{5})} = 2.5$$

5 Geometric series with, $r = \frac{1}{1 + a^2}$

$$\frac{1}{1 + a^2} < 1 \text{ provided } a \neq 0$$

$$\text{sum} = \frac{a^2}{1 - \frac{1}{1 + a^2}} = \frac{a^2(1 + a^2)}{1 + a^2 - 1} = 1 + a^2$$

6 $y = \frac{x^3 - 2x^2 + 5}{x^2 - x^3}$

a $y = -1$

b $\frac{x^3 - 2x^2 + 5}{x^2 - x^3} = -1$

$$x^3 - 2x^2 + 5 = -x^2 + x^3$$

$$5 = x^2$$

$$x = \pm\sqrt{5}$$

$$(\sqrt{5}, -1) \quad (-\sqrt{5}, -1)$$

7 $y = \frac{2x + 1}{x^2 + 1} \quad (0, 1)$

$$\frac{dy}{dx} = \frac{2(x^2 + 1) - 2x(2x + 1)}{(x^2 + 1)^2}$$

If $x = 0$, $\frac{dy}{dx} = 2$

Tangent: $y - 1 = 2x \quad y = 2x + 1$

Normal: $y - 1 = -\frac{1}{2}x \quad y = -\frac{1}{2}x + 1$

9 $x + y = -3 \quad \text{gradient} = -1$

$$y = x\sqrt{x + 1}$$

$$\frac{dy}{dx} = x \frac{1}{2}(x + 1)^{-\frac{1}{2}} + (x + 1)^{\frac{1}{2}}$$

$$= \frac{x + 2(x + 1)}{2(x + 1)^{\frac{1}{2}}}$$

$$= \frac{3x + 2}{2(x + 1)^{\frac{1}{2}}}$$

$$\frac{3x+2}{2(x+1)^{\frac{1}{2}}} = -1 \Rightarrow 3x+2 = -2(x+1)^{\frac{1}{2}}$$

$$(3x+2)^2 = 4(x+1)$$

$$9x^2 + 12x + 4 = 4x + 4$$

$$9x^2 + 8x = 0$$

$$x(9x+8) = 0$$

$$x = 0 \text{ or } -\frac{8}{9}$$

If $x = 0$, $\frac{dy}{dx} = 1$ If $x = -\frac{8}{9}$, $\frac{dy}{dx} = -1$

$\therefore y = x\sqrt{x+1}$ is parallel to the line $x + y = -3$
at $\left(-\frac{8}{9}, -\frac{8}{27}\right)$

10 $\frac{dy}{dx} = \frac{1}{2}(8x^3 - 15x^2 - 10x + 3) = -7$

normal: $y + \frac{5}{2} = \frac{1}{7}(x - 1)$

$$y = \frac{1}{7}x - \frac{37}{14}$$

$$\frac{1}{2}(2x^4 - 5x^3 - 5x^2 + 3) = \frac{1}{7}x - \frac{37}{14}$$

$$14x^4 - 35x^3 - 35x^2 + 21x = 2x - 37$$

$$14x^4 - 35x^3 - 35x^2 + 19x + 37 = 0$$

$$x = 3.0782 \quad y = -2.2031 \quad (3.08, -2.20)$$

11 $f(x) = [g(x)]^3 \therefore f(0) = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8} \quad \left(0, -\frac{1}{8}\right)$

$$f'(x) = 3[g(x)]^2 g'(x)$$

$$f'(0) = 3\left(-\frac{1}{2}\right)^2 \left(\frac{8}{3}\right) = 2$$

$$y + \frac{1}{8} = 2x \text{ or } y = 2x - \frac{1}{8}$$

12 a $y = (1 - 3x)^7 (3x + 5)^3$

$$\frac{dy}{dx} = (1 - 3x)^7 3(3x + 5)^2 3 + (3x + 5)^3$$

$$7(1 - 3x)^6 (-3)$$

$$= 9(1 - 3x)^7 (3x + 5)^2 - 21(1 - 3x)^6 (3x + 5)^3$$

$$= 3(1 - 3x)^6 (3x + 5)^2 [3(1 - 3x) - 7(3x + 5)]$$

$$= 3(1 - 3x)^6 (3x + 5)^2 (-30x - 32)$$

$$= -6(1 - 3x)^6 (3x + 5)^2 (15x + 16)$$

b $y = \sqrt{(4x^2 - 3x + 1)^5}$

$$\frac{dy}{dx} = \frac{5}{2}(4x^2 - 3x + 1)^{\frac{3}{2}}(8x - 3)$$

c $y = \frac{x^2 - 3}{\sqrt{x+1}} \quad x \neq -1$

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}} 2x - \frac{1}{2}(x+1)^{-\frac{1}{2}}(x^2 - 3)}{(x+1)}$$

$$= \frac{4x(x+1) - (x^2 - 3)}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{3x^2 + 4x + 3}{2(x+1)^{\frac{3}{2}}}$$

d $y = \sqrt{x + \sqrt{x^2 + 1}} = \left(x + (x^2 + 1)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} \left(x + (x^2 + 1)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} 2x\right)$$

$$= \frac{1 + x(x^2 + 1)^{-\frac{1}{2}}}{2 \left(x + (x^2 + 1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

$$= \frac{(x^2 + 1)^{\frac{1}{2}} + x}{2 \left(x^2 + 1\right)^{\frac{1}{2}} \left(x + (x^2 + 1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

$$= \frac{\left[(x^2 + 1)^{\frac{1}{2}} + x\right]^{\frac{1}{2}}}{2 \left(x^2 + 1\right)^{\frac{1}{2}}}$$

$$= \frac{1}{2} \sqrt{\frac{(x^2 + 1)^{\frac{1}{2}} + x}{(x^2 + 1)}}$$

e $y = (x + 2 + (x - 3)^8)^3$

$$\frac{dy}{dx} = 3(x + 2 + (x - 3)^8)^2 (1 + 8(x - 3)^7)$$

13 $f(x) = ax^3 + 6x^2 - bx$

$$f'(x) = 3ax^2 + 12x - b$$

$$f''(x) = 6ax + 12$$

$$f''(1) = 0 \therefore 6a + 12 = 0 \therefore a = -2$$

$$f'(-1) = 0 \therefore -6 - 12 - b = 0 \therefore b = -18$$

14 $y = x - 3\sqrt[3]{x} = x - 3x^{\frac{1}{3}}$

a $(0, 0) \quad x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3) = 0$

$$x = 0 \text{ or } x^{\frac{2}{3}} = 3$$

$$x = \pm 27 \quad (\sqrt[3]{27}, 0) \quad (-\sqrt[3]{27}, 0)$$

b $\frac{dy}{dx} = (1 - x^{\frac{-2}{3}})$
 $1 - x^{\frac{-2}{3}} = 0 \quad \therefore 1 = \frac{1}{x^{\frac{2}{3}}} \quad \therefore x^{\frac{2}{3}} = 1 \quad x = \pm 1$

$$\frac{d^2y}{dx^2} = \frac{2}{3} x^{\frac{-5}{3}} = \frac{2}{3\sqrt[3]{x^5}}$$

If $x = 1$, $\frac{d^2y}{dx^2} > 0 \quad \therefore$ minimum at $(1, -2)$

If $x = -1$, $\frac{d^2y}{dx^2} < 0 \quad \therefore$ maximum at $(-1, 2)$

c $\frac{d^2y}{dx^2} \neq 0 \quad \therefore$ no points of inflexion

d

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	-	0	+

i function increases for $]-\infty, -1[\cup]1, \infty[$

ii function decreases for $]-1, 1[$

15 $y = \frac{2x}{x^2 - 1}$

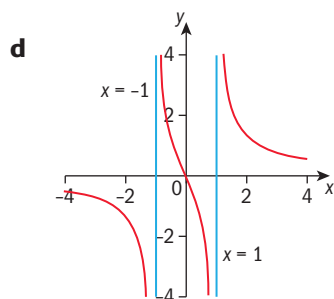
a $x = 1, x = -1, y = 0$

b $f(-x) = \frac{-2x}{(-x)^2 - 1} = \frac{-2x}{x^2 - 1} = -f(x)$

$\therefore$ function is odd

c $\frac{dy}{dx} = \frac{(x^2 - 1)2 - 2x(2x)}{(x^2 - 1)^2}$
 $= \frac{-2x^2 - 2}{(x^2 - 1)^2} < 0$

$\therefore \frac{dy}{dx} < 0$



16 $f(x) = \frac{(x-3)^2}{x^2 - 3}$

a $(0, -3) \quad (3, 0) \quad x = \pm \sqrt{3} \quad y = 1$

b $f'(x) = \frac{(x^2 - 3)2(x-3) - 2x(x-3)^2}{(x^2 - 3)^2}$
 $= \frac{2(x-3)[x^2 - 3 - x(x-3)]}{(x^2 - 3)^2}$
 $= \frac{2(x-3)(3x-3)}{(x^2 - 3)^2} = \frac{6(x-3)(x-1)}{(x^2 - 3)^2}$

$f'(x) = 0 \Rightarrow x = 3 \text{ or } 1$

$f''(x) = \frac{6x^2 - 24x + 18}{(x^2 - 3)^2}$

$f''(x) = \frac{(x^2 - 3)^2(12x - 24) - 2(x^2 - 3)2x(6x^2 - 24x + 18)}{(x^2 - 3)^4}$
 $= \frac{(x^2 - 3)(12x - 24) - 4x(6x^2 - 24x + 18)}{(x^2 - 3)^3}$
 $= \frac{-12x^3 + 72x^2 - 108x + 72}{(x^2 - 3)^3}$

If $x = 3$, $f''(x) > 0 \quad \therefore$ minimum at $(3, 0)$

If $x = 1$, $f''(x) < 0 \quad \therefore$ maximum at $(1, -2)$

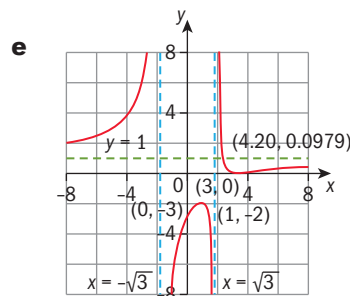
c $f''(x) = 0 \Rightarrow x = 4.1958$

x	$x < 4.1958$	$x > 4.1958$
$f''(x)$	+	-

$\therefore$ point of inflexion at $(4.20, 0.0979)$

d i increasing for $]-\infty, -\sqrt{3}[\cup]-\sqrt{3}, 1[\cup]3, \infty[$

ii decreasing for $]1, \sqrt{3}[\cup]\sqrt{3}, 3[$



17 $x = y^5 - y \quad 0 = y(y^4 - 1)$

$y = 0, \pm 1 \quad (0, 0) \quad (0, 1) \quad (0, -1)$

$\frac{dx}{dy} = 5y^4 - 1 \quad \therefore \frac{dy}{dx} = \frac{1}{5y^4 - 1}$

At $(0, 0) \quad \frac{dy}{dx} = -1$

At $(0, 1) \quad \frac{dy}{dx} = \frac{1}{4}$



Review exercise

1 $(1.5, 0) \quad (x, \sqrt{x}) \quad \text{Let } l = \text{distance}$

$l^2 = (x - 1.5)^2 + x$
 $= x^2 - 2x + 2.25$

$l = (x^2 - 2x + 2.25)^{\frac{1}{2}}$

$\frac{dl}{dx} = \frac{1}{2}(x^2 - 2x + 2.25)^{\frac{1}{2}}(2x - 2)$
 $= \frac{x - 1}{(x^2 - 2x + 2.25)^{\frac{1}{2}}}$

$\frac{dl}{dx} = 0 \Rightarrow x = 1$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	-	0	+

$\therefore$ minimum distance when $x = 1$

$$\text{minimum distance} = \sqrt{1.25}$$

- 2 Let r = radius, x = side of square

$$4\pi r + 4x = 80$$

$$x = 20 - \pi r$$

$$A = 2\pi r^2 + x^2$$

$$= 2\pi r^2 + (20 - \pi r)^2$$

$$= 2\pi r^2 + 400 - 40\pi r + \pi^2 r^2$$

$$\frac{dA}{dr} = 4\pi r - 40\pi + 2\pi^2 r$$

$$\frac{dA}{dr} = 0 \Rightarrow 4\pi r + 2\pi^2 r = 40\pi$$

$$2r + \pi r = 20$$

$$r = \frac{20}{2 + \pi}$$

$$\frac{d^2 A}{dr^2} = 4\pi + 2\pi^2 > 0 \therefore \text{minimum}$$

$$r = \frac{20}{2 + \pi}$$

- 3 $\frac{dr}{dt} = 3 \text{ cm min}^{-1}$ $\frac{dh}{dt} = -4 \text{ cm min}^{-1}$

$$v = \pi r^2 h$$

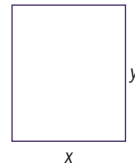
$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$

$$= \pi(81)(-4) + 2\pi(9)(12)(3)$$

$$= 324\pi \text{ cm}^3 \text{ min}^{-1}$$

increasing at a rate of $324\pi \text{ cm}^3 \text{ min}^{-1}$

4



$$xy = 180 \quad y = \frac{180}{x}$$

$$\text{printing area, } A = (x - 2)(y - 3)$$

$$A = xy - 3x - 2y + 6$$

$$= 180 - 3x - \frac{360}{x} + 6$$

$$\frac{dA}{dx} = -3 + \frac{360}{x^2}$$

$$\frac{dA}{dx} = 0 \Rightarrow 3x^2 = 360$$

$$x^2 = 120$$

$$x = \sqrt{120} = 10.95 \quad y = 16.43$$

$$\frac{d^2 A}{dx^2} = \frac{-720}{x^3} < 0 \therefore \text{maximum}$$

$\therefore$ dimensions are 11.0 cm by 16.4 cm

- 5 $\frac{dx}{dt} = \frac{1}{1 + 2x} = (1 + 2x)^{-1}$

$$\text{acceleration} = \frac{d^2 x}{dt^2} = -(1 + 2x)^{-2} 2 \frac{dx}{dt}$$

$$= \frac{-2}{(1 + 2x)^3}$$

$$\text{At } x = 2, \text{ acceleration} = \frac{-2}{125}$$

5

Aesthetics in mathematics

Answers

Skills check

1 $f(x) = \frac{x}{x-1}, x \neq 1$

inverse: $x = \frac{y}{y-1}$

$x(y-1) = y$

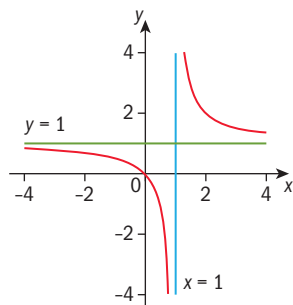
$xy - y = x$

$y(x-1) = x$

$y = \frac{x}{x-1}$

$f^{-1}(x) = \frac{x}{x-1}, x \neq 1$

$\therefore f^{-1}(x) = f(x)$



2 $f(x) = ax - b, g(x) = \frac{x+b}{a}$

$f \circ g(x) = f\left(\frac{x+b}{a}\right) = a\left(\frac{x+b}{a}\right) - b = x$

$g \circ f(x) = g(ax - b) = \frac{ax - b + b}{a} = x$

$\therefore f \circ g(x) = g \circ f(x)$

Exercise 5A

1 $u_1 = 1, u_n = \frac{u_{n+1}}{1+u_{n-1}}, n \in \mathbb{Z}^+$

$u_2 = \frac{1}{2}, u_3 = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}$

$u_4 = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{2}} = \frac{1}{4}, u_5 = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{3}} = \frac{1}{5}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, u_n = \frac{1}{n}$

2 $u_1 = 2, u_n = \frac{u_{n-1}}{1+u_{n-1}}$

$n \in \mathbb{Z}^+, n \geq 2$

$u_2 = \frac{2}{1+2} = \frac{2}{3}, u_3 = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{5}$

$u_4 = \frac{\frac{2}{5}}{1+\frac{2}{5}} = \frac{2}{7}, u_5 = \frac{\frac{2}{7}}{1+\frac{2}{7}} = \frac{2}{9}$

$u_6 = \frac{\frac{2}{9}}{1+\frac{2}{9}} = \frac{2}{11}, 2, -2, \frac{-2}{3}, \frac{-2}{5}, \frac{-2}{7}, \frac{-2}{9}, u_n = \frac{-2}{2n-3}$

3 a $u_0 = 2, u_n = u_{n-1} = \frac{1}{2}n, n \in \mathbb{Z}^+$

$u_1 = 2 - \frac{1}{2} = \frac{3}{2}, u_2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$

$u_3 = \frac{5}{4} - \frac{1}{8} = \frac{9}{8}, u_4 = \frac{9}{8} - \frac{1}{16} = \frac{17}{16}$

$u_5 = \frac{17}{16} - \frac{1}{32} = \frac{33}{32}, 2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \frac{33}{32}$

b $u_n = \frac{2^n + 1}{2^n}$

c $P(n): u_n = \frac{2^n + 1}{2^n}$

$P(0) \Rightarrow u_0 = \frac{2^0 + 1}{2^0} = \frac{2}{1} = 2$

$\therefore P(0)$ is true

Assume $P(k), u_k = \frac{2^k + 1}{2^k}$

Prove $P(k+1), u_{k+1} = u_k - \frac{1}{2^{k+1}}$

$$\begin{aligned} &= \frac{2^k + 1}{2^k} - \frac{1}{2^{k+1}} \\ &= \frac{2(2^k + 1) - 1}{2^{k+1}} \\ &= \frac{2^{k+1} + 2 - 1}{2^{k+1}} \\ &= \frac{2^{k+1} + 1}{2^{k+1}} \end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$ and $P(0)$ is true

$\therefore$ by mathematical induction, $u_n = \frac{2^n + 1}{2^n}$

4 a $u_1 = 1, u_n = u_{n-1} + 2n - 3, n \in \mathbb{Z}$

$u_2 = 1 + 4 - 3 = 2$

$u_3 = 2 + 6 - 3 = 5$

$u_4 = 5 + 8 - 3 = 10$

$u_5 = 10 + 10 - 3 = 17, 1, 2, 5, 10, 17$

- b** $P(n) : un = n^2 - 2n + 2$
 $P(1) \Rightarrow u_1 = 1^2 - 2(1) + 2 = 1 \quad \therefore P(1)$ is true
 Assume $P(k) \quad u_k = k^2 - 2k + 2$
 Prove $P(k+1) \quad u_{k+1} = u_k + 2(k+1) - 3$

$$= k^2 - 2k + 2 + 2k + 2 - 3$$

$$= k^2 + 1$$

$$(k+1)^2 - 2(k+1) + 2 = k^2 + 2k + 1 - 2k - 2 + 2$$

$$= k^2 + 1$$

$$\therefore u_{k+1} = (k+1)^2 - 2(k+1) + 2$$

$$\therefore P(k) \Rightarrow P(k+1) \text{ and } P(1) \text{ is true}$$

$$\therefore \text{by mathematical induction, } u_n = n^2 - 2n + 2$$

Exercise 5B

- 1 a** $(64)^{\frac{2}{3}} = 4^2 = 16$
b $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$
c $\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
- 2 a** $\left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = \left(\frac{8x}{b^{-3}x^{-2}}\right)^{\frac{2}{3}} = (8x^3b^3)^{\frac{2}{3}}$

$$\therefore \left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = (2xb)^2 = 4b^2x^2$$

b $\frac{a^{-1}-a^{-2}}{a^{-3}} = \frac{a^{-1}}{a^{-3}} - \frac{a^{-2}}{a^{-3}} = a^2 - a = a(a-1)$
c $\frac{x^3 \times x^{-7}}{x^{-4}} = \frac{x^{-4}}{x^{-4}} = 1$
- 3** $\sqrt{y^3} \div \sqrt[3]{y^2} = y^{\frac{3}{2}} \div y^{\frac{2}{3}} = y^{\left(\frac{3}{2}-\frac{2}{3}\right)} = y^{\frac{5}{6}}$
 when $y = 64$, $y^{\frac{5}{6}} = (\sqrt[6]{64})^5 = 2^5 = 32$
- 4** $\frac{(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z}}{(xz)^{\frac{7}{2}}} = \frac{x^8y^2z^{-6} \times x^{\frac{5}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}}$

$$= \frac{x^{\frac{11}{2}}y^{\frac{3}{2}}z^{-\frac{11}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}} = x^2y^3z^{-9}$$

$$= \frac{x^2y^3}{z^9}$$
- 5** $5 \times 4^{3n+1} - 20 \times 8^{2n} = 5 \times (2^2)^{3n+1} - 20 \times (2^3)^{2n}$

$$= 5 \times 2^{6n+2} - 20 \times 2^{6n}$$

$$= 5 \times 2^2 \times 2^{6n} - 20 \times 2^{6n}$$

$$= 20 \times 2^{6n} - 20 \times 2^{6n}$$

$$= 0$$
- 6** $4^x + 2 = 3 \times 2^x$
 $(2^x)^2 - 3(2^x) + 2 = 0$
 $(2^x - 1)(2^x - 2) = 0$
 $2^x = 1 \text{ or } 2^x = 2$
 $x = 0 \text{ or } 1$

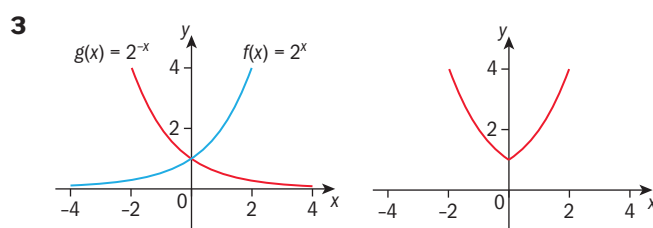
Exercise 5C

- 1** $250000(1+r)^{20} = 450000$
 $(1+r)^{20} = 1.8$
 $1+r = 1.0298$
 $r = 0.0298 = 2.98\% = 3\%$ (nearest percent)
- 2 a** $61.08 = 17.48(1+r)^7$
 $(1+r)^7 = 3.494$
 $1+r = 1.1957$
 $r = 0.196 = 19.6\%$
b $77.45 = 61.08(1+r)^4$
 $(1+r)^4 = 1.268\dots$
 $1+r = 1.0612$
 $r = 0.0612 = 6.12\%$
c $97.87 = 72.99(1+r)^{12}$
 $(1+r)^{12} = 1.3408$
 $1+r = 1.0247$
 $r = 0.0247 = 2.47\%$
- 3** Samira: After 15 years (60 quarters), she will have
 $1000 \left(1 + \frac{0.08}{4}\right)^{60} = 1000 \times (1.02)^{60}$
 $\approx \$3280$
 Hemanth: $500(1.08)^{15} + 500 \left(1 + \frac{0.084}{12}\right)^{12 \times 15}$
 $= 500(1.08)^{15} + 500(1.007)^{180}$
 $= 1586.08 + 1754.99$
 $\approx \$3340$
- 4**
- | End of year | Amount owing | Pays back |
|-------------|-----------------------------------|-------------|
| 1 | $15000 \times 1.05 = 15750$ | 10500 |
| 2 | $5250 \times 1.05 = 5512.5$ | 3675 |
| 3 | $1837.5 \times 1.05 = 1929.375$ | 1286.25 |
| 4 | $643.125 \times 1.05 = 675.28$ | 450.1875 |
| 5 | $225.09375 \times 1.05 = 236.348$ | 157.565625 |
| | | 16069.00313 |

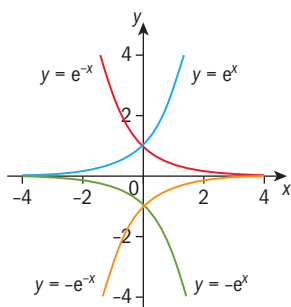
$\therefore$ Guiseppe has paid back approx. € 16700

Exercise 5D

- 1** Red curve: $(1, 2.5) \quad 2.5 = a^1 \quad \therefore a = 2.5$
 Blue curve: $(-1, 4) \quad 4 = a^{-1} \quad \therefore a = \frac{1}{4}$
- 2** $f(x) = 2^x \quad f(x+1) = 2^{(x+1)} = 2(2^x) = 2f(x)$
 $f(x+a) = 2^{x+a} = 2^a(2^x) = 2^a f(x)$



- 4 $e^x + 1 = e^{x+1}$
 $x = -0.541$
- 5 a reflection in the y -axis
 b reflection in the x -axis
 c rotation of 180° about the origin (or reflection in the y -axis followed by reflection in the x -axis)



Exercise 5E

- 1 a $5^3 = 125 \Rightarrow \log_5 125 = 3$
 b $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$
 c $27^{\frac{1}{3}} = 3 \Rightarrow \log_{27} 3 = \frac{1}{3}$
 d $10^{-3} = 0.001 \Rightarrow \log_{10} 0.001 = -3$
 e $m = n^2 \Rightarrow \log_n m = 2$
 f $a^b = 2 \Rightarrow \log_a 2 = b$
- 2 a $\log_3 9 = 2 \Rightarrow 3^2 = 9$
 b $\log_{10} 1000000 = 6 \Rightarrow 10^6 = 1000000$
 c $\log_{49} 7 = \frac{1}{2} \Rightarrow 49^{\frac{1}{2}} = 7$
 d $\log_a 1 = 0 \Rightarrow a^0 = 1$
 e $\log_4 a = b \Rightarrow 4^b = a$
 f $\log_p q = r \Rightarrow p^r = q$
- 3 a $\log_8 64 = 2$
 b $\log_9 3 = \frac{1}{2}$
 c $\log_{10} 0.01 = -2$
 d $\log_{144} 12 = \frac{1}{2}$
 e $\log_{37} 1 = 0$
 f $\log_a \sqrt[3]{a} = \frac{1}{3}$
- 4 a $\log_x 81 = 2 \therefore x^2 = 81 \therefore x = 9$
 b $\log_3 x = 4 \therefore 3^4 = x \therefore x = 81$
 c $\log_{11} 121 = x \therefore x = 2$
 d $\log_x 5 = \frac{1}{3} \therefore x^{\frac{1}{3}} = 5 \therefore x = 125$
 e $\log_x 16 = \frac{2}{3} \therefore x^{\frac{2}{3}} = 16 \therefore x = 16^{\frac{3}{2}} = 4^3 = 64$
 f $\log_x 32 = -5 \therefore x^{-5} = 32 \therefore \frac{1}{x^5} = 32, x^5 = \frac{1}{32}, x = \frac{1}{2}$

Exercise 5F

- 1 a $\log_a \frac{p^2}{q} = 2 \log_a p - \log_a q$
 b $\log_a \sqrt[3]{\frac{p}{q^2}} = \frac{1}{3} \log_a \frac{p}{q^2} = \frac{1}{3} \log_a p - \frac{2}{3} \log_a q$

- 2 a $\log 4 + 2 \log 3 - \log 6 = \log \frac{4 \times 9}{6} = \log 6$
 b $\frac{1}{2} \log_a p + \frac{1}{4} \log_a q^2 = \log_a p^{\frac{1}{2}} + \log_a q^{\frac{1}{2}} = \log_a \sqrt{pq}$
 c $2 - \log 5 = \log 100 - \log 5 = \log 20$
- 3 a $\log 5 + \log 8 - \log 4 = \log \frac{5 \times 8}{4} = \log 10 = 1$
 b $\log_2 48 - \frac{1}{3} \log_2 27 = \log_2 48 - \log_2 3 = \log_2 16 = 4$
 c $2 + \log_5 10 - \log_5 2 = 2 + \log_5 5 = 2 + 1 = 3$
- 4 a $3 \log y = 2 \log x \Rightarrow y^3 = x^2 \therefore y = x^{\frac{2}{3}}$
 b $\log y = \log x + \log 2 \Rightarrow y = 2x$
 c $\log y - 3 \log x = \log 2 \Rightarrow \frac{y}{x^3} = 2 \therefore y = 2x^3$
 d $\log y = 2 + 3x \Rightarrow y = 10^{2+3x}$

Exercise 5G

- 1 a $\log_3 2 \times \log_3 81 = \log_3 2 \times \frac{\log_3 81}{\log_3 2} = 4$
 b $\log_6 10 \times \log 6 = \log_6 10 \times \frac{\log_6 6}{\log_6 10} = 1$
 c $\log_{125} 8 \times \log_8 5 = \log_{125} 8 \times \frac{\log_{125} 5}{\log_{125} 8} = \frac{1}{3}$
 d $\frac{1}{\log_2 6} + \frac{1}{\log_3 6} = \frac{1}{\log_2 6} + \frac{\log_2 3}{\log_2 6} = \frac{\log_2 2 + \log_2 3}{\log_2 6} = \frac{\log_2 6}{\log_2 6} = 1$
 e $\frac{1}{\log_4 6} + \frac{1}{\log_9 6} = \frac{1}{\log_4 6} + \frac{\log_4 9}{\log_4 6} = \frac{\log_4 4 + \log_4 9}{\log_4 6} = \frac{\log_4 36}{\log_4 6} = \frac{2 \log_4 6}{\log_4 6} = 2$
 f $\log_5 40 - \frac{1}{\log_8 5} = \log_5 40 - \frac{\log_5 8}{\log_5 5} = \log_5 5 = 1$
- 2 a let $a^{\log b} = x \Rightarrow \log_a x = \log b$
 $\frac{\log x}{\log a} = \log b$
 $\frac{\log x}{\log b} = \log a$
 $\log_b x = \log a$
 $\therefore x = b^{\log a}$
 $\therefore a^{\log b} = b^{\log a}$
 b $\frac{1}{\log_a ab} + \frac{1}{\log_b ab} = \frac{\log_{ab} a}{\log_{ab} ab} + \frac{\log_{ab} b}{\log_{ab} ab} = \log_{ab} a^a + \log_{ab} b^b = \log_{ab} a^{(ab)} = 1$
 $\therefore \frac{1}{\log_a ab} + \frac{1}{\log_b ab} = 1$

- 3 $p = \log a^x \quad q = \log a^y$
 $\log_x a = \frac{1}{\log_a x} = \frac{1}{p} = \log_y a = \frac{1}{\log_a y} = \frac{1}{q}$
 a $\log_{xy} a = \frac{\log_a a}{\log_a xy} = \frac{1}{\log_a x + \log_a y} = \frac{1}{p+q}$
 b $\log_{xy} a = \frac{1}{\log_a \left(\frac{x}{y}\right)} = \frac{1}{\log_a x - \log_a y} = \frac{1}{p-q}$

Exercise 5H

- 1 a** $5^x = 7$
 $x \log 5 = \log 7$
 $x = \frac{\log 7}{\log 5}$
 $x = 1.21$
- b** $4^{2x-1} = 3$
 $(2x-1)\log 4 = \log 3$
 $2x-1 = \frac{\log 3}{\log 4}$
 $x = \frac{1}{2} \left(\frac{\log 3}{\log 4} + 1 \right)$
 $\therefore x = 0.896$
- 2** $(2^x)(5^x) = 0.01$
 $10^x = 0.01$
 $\therefore x = -2$

Exercise 5I

- 1 a** $2^{3x} = 5$
 $3x \log 2 = \log 5$
 $x = \frac{\log 5}{3 \log 2}$
 $x = 0.774$
- b** $3^x (3^{x-1}) = 10$
 $3^{2x-1} = 10$
 $(2x-1) \log 3 = \log 10$
 $2x-1 = \frac{1}{\log 3}$
 $x = \frac{1}{2} \left(\frac{1}{\log 3} + 1 \right)$
 $x = 1.55$
- 2 a** $4 \log_3 x = \log_x 3$
 $4 \log_3 x = \frac{1}{\log_3 x}$
 $(\log_3 x)^2 = \frac{1}{4} \therefore \log_3 x = \pm \frac{1}{2}$
 $x = 3^{\frac{1}{2}} \text{ or } x = 3^{-\frac{1}{2}}$
- b** $3 \log_2 x + \log_2 27 = 3$
 $\log_2 (27x^3) = 3$
 $\therefore 27x^3 = 8$
 $x^3 = \frac{8}{27}$
 $x = \frac{2}{3}$
- 3** $9^x - 6(3^x) - 16 = 0$
 $(3^x)^2 - 6(3^x) - 16 = 0$
 $(3^x - 8)(3^x + 2) = 0$
 $3^x = 8$
 $x \log 3 = \log 8, x = 1.89$

- 4** $\log_4 x + 12 \log_x 4 - 7 = 0$
 $\log_4 x + \frac{12}{\log_4 x} - 7 = 0$
 $(\log_4 x)^7 - 7 \log_4 x + 12 = 0$
 $(\log_4 x - 3)(\log_4 x - 4) = 0$
 $\log_4 x = 3 \text{ or } \log_4 x = 4$
 $x = 4^3 \text{ or } x = 4^4$
 $x = 64 \text{ or } 256$
- 5** $5^{x+1} + \frac{4}{5^x} - 21 = 0$
 $5(5^x)^2 - 21(5^x) + 4 = 0$
 $(5(5^x) - 1)(5^x - 4) = 0$
 $5^x = \frac{1}{5} \text{ or } 5^x = 4$
 $x = -1 \text{ or } x \log 5 = \log 4$
 $x = -1 \text{ or } 0.861$
- 6** $\log_3 x + \log_x 9 - 3 = 0$
 $\log_3 x + \frac{\log_3 9}{\log_3 x} - 3 = 0$
 $(\log_3 x)^2 - 3 \log_3 x - 2 = 0$
 $(\log_3 x - 2)(\log_3 x - 1) = 0$
 $\log_3 x = 2 \text{ or } \log_3 x = 1$
 $x = 3^2 \text{ or } x = 3^1$
 $x = 3 \text{ or } 9$
- 7** $3 \times 9^x - 2 \times 4^x = 5 \times 6^x$
 $3(3^x)^2 - 5(2^x)(3^x) - 2(2^x)^2 = 0$
 $(3(3^x) + 2^x)(3^x - 2(2^x)) = 0$
 $3(3^x) = -2^x \text{ or } 3^x = 2(2^x)$
 $3(1.5)^x = -1 \text{ or } 1.5^x = 2$
 No solution, $x \log 1.5 = \log 2$
 $x = 1.71$
- 8** $6 \log_2 x + 6 \log_8 y = 7$
 $6 \log_2 x + \frac{6 \log_2 y}{\log_2 8} = 7$
 $6 \log_2 x + 2 \log_2 y = 7$
 $\log_2 x^6 y^2 = 7$
 $x^6 y^2 = 2^7 \therefore x^6 y^2 = 128 \quad (1)$
 $4 \log_4 x + 4 \log_2 y = 9$
 $\frac{4 \log_2 x}{\log_2 4} + 4 \log_2 y = 9$
 $2 \log_2 x + 4 \log_2 y = 9$
 $\log_2 x^2 y^4 = 9$
 $\therefore x^2 y^4 = 2^9 \therefore x^2 y^2 = 512 \quad (2)$
 from (1) $y^2 = \frac{128}{x^6} \therefore x^2 \left(\frac{128}{x^6} \right)^2 = 512$
 $16384 = 512 x^{10} \therefore x^{10} = 32$
 $x = \sqrt[10]{32}$
 $y^2 = \frac{128}{x^6} = 16 \therefore y = 4$

$$\begin{aligned} 9 \quad 2 \log xy = 1 &\Rightarrow x = y^2 \\ xy = 125 &\therefore y^3 = 125 \\ y = 5, x = 25 \end{aligned}$$

$$\begin{aligned} 10 \quad y \log_2 8 = x &\Rightarrow 3y = x \\ 2^x + 8^y &= 64 \\ 2^{3y} + 2^{3y} &= 64 \\ 2^{3y+1} &= 26 \end{aligned}$$

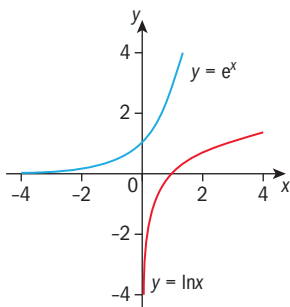
$$\begin{aligned} 3y + 1 &= 6 \\ y &= \frac{5}{3}, \quad x = 5 \end{aligned}$$

$$\begin{aligned} 11 \quad a \quad \log_5 x = y &= \log_{25}(2x - 1) \\ x = 5^y \quad 25^y &= 2x - 1 \\ (5^y)^2 &= 2x - 1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ x = 1, y &= 0 \end{aligned}$$

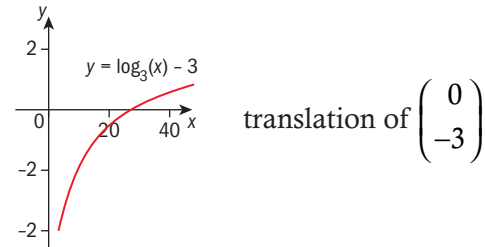
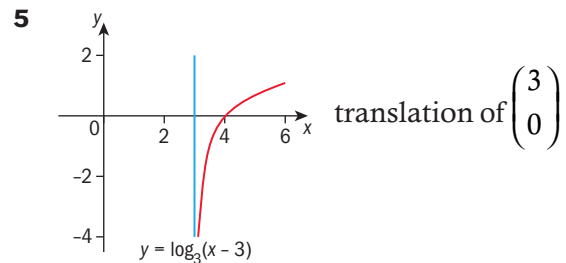
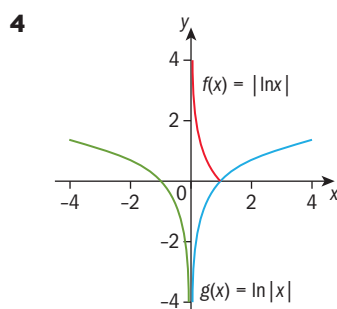
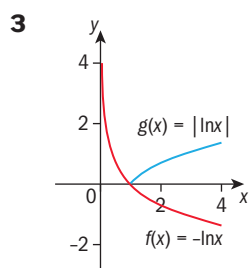
$$\begin{aligned} b \quad \log(x+y) = 0 &\Rightarrow x+y = 1 \Rightarrow y = 1-x \\ 2 \log x = \log(y+5) &\Rightarrow x^2 = y+5 \\ x^2 &= 1-x+5 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x = 2 \quad y &= -1 \\ (x \text{ cannot be negative}) \end{aligned}$$

Exercise 5J

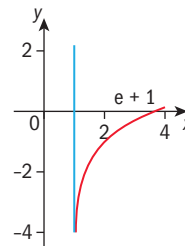
$$\begin{aligned} 1 \quad f(x) &= e^x : \text{domain is } x \in \mathbb{R} \\ \text{range is } y &\in \mathbb{R}, y > 0 \\ f^{-1}(x) &= \ln x : \text{domain is } x \in \mathbb{R}, x > 0 \\ \text{range is } y &\in \mathbb{R} \end{aligned}$$



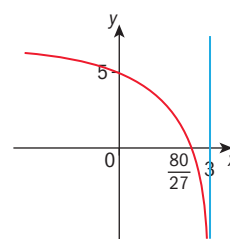
$$\begin{aligned} 2 \quad f(x) &= a^x \quad y = a^x \\ \text{Inverse : } x &= a^y, \quad y = \log_a x \\ \therefore f^{-1}(x) &= \log_a x \\ f \circ f^{-1}(x) &= f(\log_a x) = a^{\log_a x} \\ \therefore a^{\log_a x} &= x \end{aligned}$$



$$\begin{aligned} 6 \quad a \quad y &= \ln(x-1) - 1 \\ x-1 > 0 &\therefore x > 1 \\ \text{domain : } x &\in \mathbb{R}, x > 1 \\ \text{asymptote : } x &= 1 \\ x\text{-intercept } 0 &= \ln(x-1) - 1 \\ \ln(x-1) &= 1 \\ (e+1, 0) \quad x-1 &= e \\ x &= e+1 \end{aligned}$$



$$\begin{aligned} b \quad y &= \log_3(9-3x) + 2 \\ 9-3x &> 0 \\ 9 &> 3x \\ x &< 3 \\ \text{domain : } x &\in \mathbb{R}, x < 3 \\ \text{asymptote : } x &= 3 \\ x\text{-intercept } 0 &= \log_3(9-3x) + 2 \\ -2 &= \log_3(9-3x) \\ 3^{-2} &= 9-3x \\ 3x &= 9 - \frac{1}{9} = \frac{80}{9} \\ x &= \frac{80}{27} \quad \left(\frac{80}{27}, 0\right) \\ y\text{-intercept } y &= \log_3 9 + 2 = 4(0, 4) \end{aligned}$$



Exercise 5K

- 1 a** $y = \frac{3}{2} e^{x^2} \quad \frac{d^2 y}{dx^2} = \frac{3}{2} (2xe^{x^2}) = 3xe^{x^2}$
- b** $y = \frac{-5}{e^{3x-1}} = -5e^{-(3x-1)} \quad \frac{dy}{dx} = 15e^{-(3x-1)} = \frac{15}{e^{3x-1}}$
- c** $y = e^{4x-1} + 4 \quad \frac{dy}{dx} = 4e^{4x-1}$
- d** $y = e^x + \frac{1}{e^x} = e^x + e^{-x} \quad \frac{dy}{dx} = e^x - e^{-x} = e^x - \frac{1}{e^x}$
- e** $y = e^{-(1-3x)} \quad \frac{dy}{dx} = 3e^{-(1-3x)}$
- f** $y = 2e^{\sqrt{x}} \quad \frac{dy}{dx} = 2 \left(\frac{1}{2} x^{-\frac{1}{2}} \right) e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$
- 2 a** $y = xe^x \quad \frac{dy}{dx} = e^x + xe^x$
- b** $y = \frac{x^2}{e^x} = x^2 e^{-x} \quad \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = \frac{2x - x^2}{e^x}$
- c** $y = \frac{e^{2x}}{\sqrt{x}} = e^{2x} x^{-\frac{1}{2}} \quad \frac{dy}{dx} = 2e^{2x} x^{-\frac{1}{2}} - e^{2x} \frac{1}{2} x^{-\frac{3}{2}}$
 $= \frac{2e^{2x}}{x^{\frac{1}{2}}} - \frac{e^{2x}}{2x^{\frac{3}{2}}} = \frac{4xe^{2x} - e^{2x}}{2x^{\frac{3}{2}}}$
- d** $y = \sqrt{x} e^{\sqrt{x}} \quad \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} e^{\sqrt{x}} + \sqrt{x} \frac{1}{2} x^{-\frac{1}{2}} e^{\sqrt{x}}$
 $= \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2} e^{\sqrt{x}}$
- 3 a** $y = \frac{e^{2x}}{\sqrt{x}} \quad \frac{dy}{dx} = 2\sqrt{x} e^{2x} - e^{2x} \frac{1}{2} x^{-\frac{1}{2}} = \frac{4xe^{2x} - e^{2x}}{2x^{\frac{3}{2}}}$
- b** $y = \frac{1-x^2}{e^x} \quad \frac{dy}{dx} = \frac{e^x(-2x) - (1-x^2)e^x}{e^{2x}} = \frac{x^2 - 2x - 1}{e^x}$
- c** $y = \frac{e^{3x}}{1+x} \quad \frac{dy}{dx} = \frac{(1+x)3e^{3x} - e^{3x}}{(1+x)^2} = \frac{e^{3x}(2+3x)}{(1+x)^2}$
- d** $y = \frac{1+e^x}{1-e^x} \quad \frac{dy}{dx} = \frac{(1-e^x)e^x - (1+e^x)(-e^x)}{(1-e^x)^2} = \frac{2e^x}{(1-e^x)^2}$
- 4 a** $y = \frac{xe^x}{1+e^x}$
 $\frac{dy}{dx} = \frac{(1+e^x)(xe^x + e^x) - xe^x(e^x)}{(1+e^x)^2} = \frac{xe^x + e^x + e^{2x}}{(1+e^x)^2}$
 $= \frac{e^x(x+1+e^x)}{(1+e^x)^2}$
- b** $y = (1+e^x)^2 \quad \frac{dy}{dx} = 2e^x(1+e^x)$
- c** $y = \sqrt{1+e^{-x}} = (1+e^{-x})^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{1}{2}(1+e^{-x})^{-\frac{1}{2}} (-e^{-x}) = \frac{-e^{-x}}{2\sqrt{1+e^{-x}}}$
- d** $y = \frac{x+e^x}{e^{-x}} = e^x(x+e^x)$
 $\frac{dy}{dx} = e^x(1+e^x) + e^x(x+e^x)$
 $= e^x(1+x+2e^x)$

e $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
 $\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$
 $= \frac{-4}{(e^x - e^{-x})^2}$

5 $f(x) = xe^x \quad -3 \leq x \leq 3$

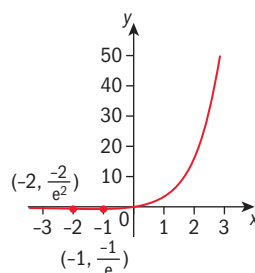
a $f'(x) = xe^x + e^x$
 $e^x(x+1) = 0 \quad \therefore x = -1 \quad y = -e^{-1}$
 $\therefore$ one stationary point at $\left(-1, -\frac{1}{e}\right)$
 $f''(x) = e^x + e^x(x+1)$
 $f''(-1) = e^{-1} > 0$ minimum

b For point of inflexion $f''(x) = 0$
 $\therefore e^x(x+2) = 0 \quad \therefore x = -2 \quad y = -2e^{-2}$

x	-3	-2	-1
$f''(x)$	$-e^{-3} < 0$	0	$e^{-1} > 0$

$\therefore$ change of sign

$\therefore$ point of inflexion at $\left(-2, -\frac{2}{e^2}\right)$



c At point of inflexion $\left(-2, -\frac{2}{e^2}\right)$
 $f'(-2) = -e^{-2} = -\frac{1}{e^2}$
 Equation of tangent: $y + \left(-2, -\frac{2}{e^2}\right) = -\frac{1}{e^2}(x+2)$
 $y = -\frac{1}{e^2}(x+4)$

d $y = 0, x = -4 \quad (-4, 0)$

e y -intercept = $\left(0, \frac{4}{e^2}\right)$ area = $\frac{1}{2} \times 4 \times \frac{4}{e^2} = \frac{8}{e^2}$

Exercise 5L

1 a $y = 5^{3x} \quad \frac{dy}{dx} = (3 \ln 5) 5^{3x}$

b $y = \ln(4x+1) \quad \frac{dy}{dx} = \frac{4}{4x+1}$

2 a $y = 1 + 2 \ln x \quad \frac{dy}{dx} = \frac{2}{x}$

b $y = \frac{1}{\ln x} = (\ln x)^{-1} \quad \frac{dy}{dx} = -(\ln x)^{-2} \frac{1}{x} = \frac{-1}{x(\ln x)^2}$

Exercise 5M

1 a $y = x^2 \ln x$ $\frac{dy}{dx} = x^2 \left(\frac{1}{x} \right) + 2x \ln x = x + 2x \ln x$

b $y = xa^x$ $\frac{dy}{dx} = (x \ln a)a^x + a^x = a^x(x \ln a + 1)$

2 a $y = \ln \left(\frac{1}{x} \right) = -\ln x$ $\frac{dy}{dx} = -\frac{1}{x}$

b $y = \ln x^2 = 2 \ln x$ $\frac{dy}{dx} = \frac{2}{x}$

c $y = \frac{\ln x}{x}$ $\frac{dy}{dx} = \frac{x \left(\frac{1}{x} \right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

3 a $x^y = e^x$
 $y \ln x = x$

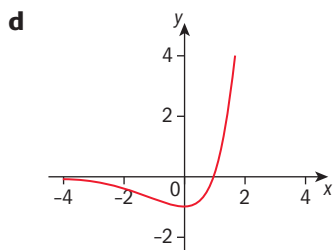
$y = \frac{x}{\ln x}$ $\frac{dy}{dx} = \frac{\ln x - x \left(\frac{1}{x} \right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

b $y = x^{2x}$
 $\ln y = 2x \ln x$
 $\frac{1}{y} \frac{dy}{dx} = 2x \left(\frac{1}{x} \right) + 2 \ln x$
 $\frac{dy}{dx} = x^{2x} (2 + 2 \ln x)$

4 a $y = e^x (x - 1)$
 $\frac{dy}{dx} = e^x + e^x (x - 1) = xe^x$
 $\frac{dy}{dx} = 0$ if $x = 0$ $\therefore$ only one stationary point

b $\frac{d^2y}{dx^2} = xe^x + e^x$
 if $x = 0$, $\frac{d^2y}{dx^2} = 1 > 0$ $\therefore$ minimum at $(0, -1)$

c $(1, 0)$



5 a $x \in \mathbb{R}$

b $f(-x) = \ln(1 + (-x)^2)$
 $= \ln(1 + x^2) = f(x)$
 $\therefore$ the y -axis is a line of symmetry

c $f(x) = \ln(1 + x^2)$
 $f'(x) = \frac{2x}{1 + x^2} = 0$ if $x = 0$
 $f''(x) = \frac{(1 + x^2)2 - 2x(2x)}{(1 + x^2)^2} = \frac{2 - 2x^2}{(1 + x^2)^2}$

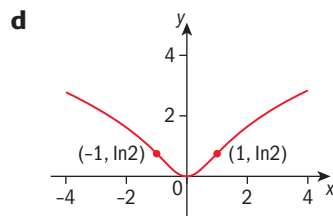
$f''(0) = 2 > 0$ $\therefore$ minimum at $(0, 0)$

$f''(x) = 0 \Rightarrow x = \pm 1$

x	-2	-1	-0	1	2
$f'(x)$	$-\frac{6}{25} < 0$	0	$2 > 0$	0	$0 - \frac{6}{25} < 0$

$\therefore$ change of sign

$\therefore$ point of inflexion at $(-1, \ln 2)$ and $(1, \ln 2)$

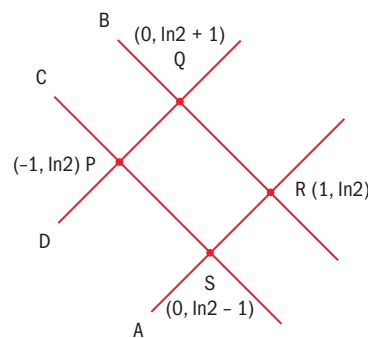


e At $(1, \ln 2)$, $f'(1) = 1$
 tangent: $y - \ln 2 = x - 1$
 $y = x - 1 + \ln 2$ (A)

normal: $y - \ln 2 = -(x - 1)$
 $y = -x + 1 + \ln 2$ (B)

At $(-1, \ln 2)$, $f'(-1) = -1$
 tangent: $y - \ln 2 = -(x + 1)$
 $y = -x - 1 + \ln 2$ (C)

normal: $y - \ln 2 = x + 1$
 $y = x + 1 + \ln 2$ (D)



Angles are 90° since gradients are ± 1

A and C intersection:

$x - 1 + \ln 2 = -x - 1 + \ln 2$

$2x = 0$

$x = 0$ $(0, \ln 2 - 1)$

B and D intersection:

$-x + 1 + \ln 2 = x + 1 + \ln 2$

$x = 0$ $(0, \ln 2 + 1)$

$\vec{PQ} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{QR} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{RS} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$\vec{SP} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore PQ = QR = RS = SP = \sqrt{2}$

$\therefore$ PQRS is a square, area 2

Exercise 5N

1 a arc length $= r\theta = \frac{5\pi}{4}$,

sector area $= \frac{1}{2} r^2 \theta = \frac{25\pi}{8}$

b arc length $= \frac{4 \times 5\pi}{12} = \frac{5\pi}{3}$,

sector area $= \frac{1}{2} \times 16 \times \frac{5\pi}{12} = \frac{10\pi}{3}$

- c** arc length = $5.4(2\pi - 1.3) \approx 26.9$ cm
sector area = $\frac{1}{2} \times 5.4^2(2\pi - 1.3) \approx 72.7$ cm²
- 2** Area = OPQ - OAB = $\frac{1}{2} \times 9^2 \times 0.8 - \frac{1}{2} \times 5^2 \times 0.8$
= $0.4(81 - 25) = 22.4$ m²
Perimeter = $5 \times 0.8 + 9 \times 0.8 + 4 + 4 = 19.2$ m
- 3** Length = AB + AC + BC + $3 \times \frac{1}{3}$ perimeter of circle
= $7.5 + 7.5 + 7.5 + 2\pi \times 7.5$
 ≈ 69.6 cm
- 4 a** Arc length = 5000 stadia
 $\therefore r \times \frac{7.2\pi}{180} = 5000 \therefore r = \frac{5000 \times 180}{7.2\pi}$ stadia
circumference = $2\pi r = 2\pi \times \frac{5000 \times 180}{7.2\pi}$ stadia
= $2 \times \frac{5000 \times 180}{7.2} \times 185$ m
= 46250 km
- b** Error = $\frac{46250 - 40008}{40008} \times 100\% = 13.5\%$
- 5** Let AB = x and BC = y , so AC = $\sqrt{x^2 + y^2}$
Crescent APBA + BQCB = semicircle ABA
+ semicircle BCB
- (semicircle ACA - ΔABC)
= $\frac{\pi x^2}{2} + \frac{\pi y^2}{2} - \frac{\pi}{2}(x^2 + y^2) + \Delta ABC$
= $0 + \Delta ABC$
= ΔABC



Review exercise

- 1** $\frac{9^{2n+2} \times 6^{2n-3}}{3^{5n} \times 6 \times 4^{n-2}} = \frac{3^{4n+4} \times 6^{2n-4}}{3^{5n} \times 2^{2n-4}} = \frac{3^{4n+4} \times 3^{2n-4}}{3^{5n}} = \frac{3^{6n}}{3^{5n}} = 3^n$
- 2** $\frac{8^{\frac{2}{3}} + 4^{\frac{3}{2}}}{16^{\frac{3}{4}}} = \frac{4+8}{8} = \frac{3}{2}$
- 3 a** $9^x - 12(3^x) + 27 = 0$
 $(3^x)^2 - 12(3^x) + 27 = 0$
 $(3^x - 9)(3^x - 3) = 0$
 $3^x = 9$ or $3^x = 3$
 $x = 1$ or $x = 2$
- b** $3^x - \frac{9}{3^x} = 8$
 $(3^x)^2 - 8(3^x) - 9 = 0$
 $(3^x - 9)(3^x + 1) = 0$
 $3^x = 9$ or $3^x = -1$
 $x = 2$
- 4 a** $\log_a x + \log_a 3 - \log_a 7 = \log_a 12$
 $\log_a \frac{3x}{7} = \log_a 12$
 $3x = 84$
 $x = 28$
- b** $\log_4 x - \log_4 5 = \frac{5}{2}$
 $\log_4 \frac{x}{5} = \frac{5}{2}$
 $4^{\frac{5}{2}} = \frac{x}{5}$
 $\frac{x}{5} = 32 \quad x = 160$
- c** $\log_3 x - \frac{6}{\log_3 x} = 1$
 $(\log_3 x)^2 - \log_3 x - 6 = 0$
 $(\log_3 x - 3)(\log_3 x + 2) = 0$
 $\log_3 x = 3$ or $\log_3 x = -2$
 $x = 3^3$ or $x = 3^{-2}$
 $x = 27$ or $\frac{1}{9}$
- d** $\log_7 x + 2\log_7 7^x = 3$
 $\log_7 x + \frac{2}{\log_7 x} = 3$
 $(\log_7 x)^2 - 3\log_7 x + 2 = 0$
 $(\log_7 x - 1)(\log_7 x - 2) = 0$
 $\log_7 x = 1$ or 2 , so $x = 7$ or 49
- 5 a** $xy = 81 \quad 3\log_x y = 1 \therefore x = y^3$
 $y^4 = 81 \quad \therefore y = 3, x = 27$
- b** $y \log_2 4 = x \quad 2^x + 4^y = 512$
 $2y = x \quad 2^x + 2^{2y} = 512$
 $2^x + 2^x = 512$
 $2^x = 256$
 $\therefore x = 8 \quad y = 4$
- c** $\ln 8 + \ln(x - 6) = 2\ln y \quad 2y - x = 2$
 $\ln 8(x - 6) = \ln y^2 \quad x = 2y - 2$
 $8(x - 6) = y^2$
 $8(2y - 8) = y^2$
 $y^2 - 16y + 64 = 0$
 $(y - 8)^2 = 0 \quad \therefore y = 8 \quad x = 14$
- 6 a** $\log y + \log \frac{1}{y} = \log 1 = 0$
- b** $\frac{\log x^5 - \log x^2}{3 \log x + \log \sqrt{x}} = \frac{3 \log x}{3.5 \log x} = \frac{6}{7}$
- c** $\ln(\ln x^2) - \ln(\ln x) = \ln\left(\frac{\ln x^2}{\ln x}\right) = \ln 2$
- 7** $\log_2 x + \log_2 x^2 + \log_2 x^3 + \dots + \log_2 x^m = 3m(m+1)$
 $\log_2 x(1 + 2 + 3 + \dots + m) = 3m(m+1)$
 $\log_2 x \left(\frac{m}{2}(m+1)\right) = 3m(m+1)$
 $\log_2 x = 6$
 $x = 2^6 = 64$
- 8** $y = 5e^{2x} + 8e^{-2x} \quad \frac{dy}{dx} = 10e^{2x} - 16e^{-2x}$
 $\frac{d^2y}{dx^2} = 20e^{2x} + 32e^{-2x} = 4(5e^{2x} + 8e^{-2x})$
= $4y$

9 $y = e^{3x}(2 + 5x)$

$$\frac{dy}{dx} = e^{3x}(5) + 3e^{3x}(2 + 5x) = e^{3x}(11 + 15x)$$

$$\frac{dy}{dx} = e^{3x}(15) + 3e^{3x}(11 + 15x) = e^{3x}(48 + 45x)$$

$$\begin{aligned} \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y &= e^{3x}(48 + 45x - 6(11 + 15x) \\ &\quad + 9(2 + 5x)) \\ &= e^{3x}(48 + 45x - 66 - 90x + 18 + 45x) \\ &= 0 \end{aligned}$$

10 $e^x - e^{-x} = 4$

$$(e^x)^2 - 4e^x - 1 = 0$$

$$e^x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$$e^x > 0 \therefore e^x = 2 + \sqrt{5} \quad \therefore x = \ln(2 + \sqrt{5})$$

$$\begin{aligned} ex + e^{-x} &= 2 + \sqrt{5} + \frac{1}{2 + \sqrt{5}} \\ &= \frac{(2 + \sqrt{5})^2 + 1}{2 + \sqrt{5}} = \frac{4 + 4\sqrt{5} + 5 + 1}{2 + \sqrt{5}} \\ &= \frac{4\sqrt{5} + 10}{2 + \sqrt{5}} = \frac{2\sqrt{5}(2 + \sqrt{5})}{2 + \sqrt{5}} \end{aligned}$$

$$\sqrt{5}e^x + e^{-x} = 2\sqrt{5}$$

11 $f(x) = \frac{4e^x}{(e^x + 1)^2}$

$$f'(x) = \frac{(e^x + 1)^2 4e^x - 4e^x 2(e^x + 1)e^x}{(e^x + 1)^4}$$

$$= \frac{4e^x(e^x + 1)^2 - 8e^{2x}}{(e^x + 1)^3}$$

$$= \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$$f'(0) = 0 \therefore \text{stationary point at } (0, 1)$$

$$f'(x) = \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$$\begin{aligned} \Rightarrow f''(x) &= \frac{(e^x + 1)^3(4e^x - 8e^{2x}) - 3(e^x + 1)^2 e^x(4e^x - 4e^{2x})}{(e^x + 1)^6} \\ &= \frac{4e^{3x} - 16e^{2x} + 4e^x}{(e^x + 1)^4} \end{aligned}$$

$$\therefore f''(0) = \frac{4 - 16 + 4}{16} > 0 \text{ so } (0, 1) \text{ is a maximum point}$$

$$\text{Points of inflexion: } f''(0) = 0$$

$$\Rightarrow 4e^{3x} - 16e^{2x} + 4e^x = 0$$

$$\Rightarrow 4e^{2x} - 16e^x + 4 = 0$$

$$\Rightarrow e^{2x} - 4e^x + 1 = 0$$

$$\Rightarrow e^x = \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$\Rightarrow x = \ln(2 \pm \sqrt{3})$$

12 a $f(x) = (\ln x)^2 \quad x \in \mathbb{R}, x > 0$

b $f'(x) = \frac{2 \ln x}{x}$

$$f''(x) = x\left(\frac{2}{x}\right) - 2 \ln x = \frac{2 - 2 \ln x}{x^2}$$

c $f'(x) = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1 \quad (1, 0)$

$$f''(1) = 2 > 0 \therefore \text{minimum at } (1, 0)$$

$$f''(x) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e \quad (e, 1)$$

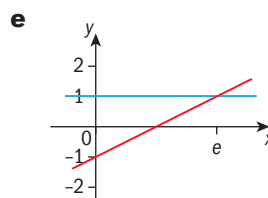
x	2	e	3
f''(x)	0.153 > 0	0	-0.0219 < 0

change of sign

$\therefore$ point of inflexion at $(e, 1)$

d $f'(e) = \frac{2}{e} \quad y - 1 = \frac{2}{e}(x - e)$

$$y = \frac{2}{e}x - 1$$



$$1 = \frac{2}{e}x - 1$$

$$2 = \frac{2}{e}x$$

$$\therefore x = e$$

$$\text{area} = \frac{1}{2}(2)(e) = e$$

13 Radius of sector = $AN = a\sqrt{3}$,

$$\text{area of sector} = \frac{1}{2}$$

$$r^2 \theta = \frac{1}{2}a^2 \times 3 \times \frac{\pi}{3}$$

$$= \frac{\pi a^2}{2}$$

$$\therefore S_1 = \frac{\pi a^2}{2} - \text{area } \triangle ADE$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}r^2 \sin 60^\circ$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}3a^2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{\pi a^2}{2} - \frac{3a^2\sqrt{3}}{4} = \frac{a^2(2\pi - 3\sqrt{3})}{4}$$

$$\frac{AE}{AC} = \frac{r}{2a} = \frac{\sqrt{3}}{2}, \text{ so } \frac{AM}{AN} = \frac{\sqrt{3}}{2}$$

$$\therefore AM = \frac{\sqrt{3}}{2} \times a\sqrt{3} = \frac{3a}{2}$$

$$\therefore S_2 = \frac{1}{2}AM^2 \theta - \triangle AFG$$

$$= \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\pi}{3} - \frac{1}{2}AM^2 \sin 60^\circ$$

$$= \frac{3\pi a^2}{8} - \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{a^2(6\pi - 9\sqrt{3})}{16} = \frac{3a^2(2\pi - 3\sqrt{3})}{16}$$

$$\begin{aligned}\text{Similarly } S_3 &= \frac{1}{2} AC^2 \theta - \frac{1}{2} AC^2 \sin 60^\circ \\ &= \frac{1}{2} \left(\frac{3a\sqrt{3}}{4} \right)^2 \frac{\pi}{2} - \frac{1}{2} \left(\frac{3a\sqrt{3}}{4} \right)^2 \frac{\sqrt{3}}{2} \\ \therefore S_3 &= \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\pi}{3} - \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\sqrt{3}}{2} \\ &= \frac{a^2}{64} (18\pi - 27\sqrt{3}) = \frac{9a^2}{64} (2\pi - 3\sqrt{3})\end{aligned}$$

Hence S_1, S_2, S_3 form a geometric series

with $r = \frac{3}{4}$.

$$\text{Total area} = \frac{a}{1-r} = \frac{a^2}{4} (2\pi - 3\sqrt{3}) = a^2 (2\pi - 3\sqrt{3})$$

$$\begin{aligned}\mathbf{14} \text{ Area required} &= \text{area of regular hexagon of side} \\ &\quad 8 \text{ cm} - 6 \times \text{area of sector of circle} \\ &\quad \text{of angle } 120^\circ - \text{central circle} \\ &= 6 \times \text{area of equilateral triangle} - 6 \times \frac{\pi r^2}{3} \\ &\quad - \pi r^2 \quad (r = 4) \\ &= 6 \times \frac{1}{2} \times 2r \times 2r \sin 60^\circ - 3\pi r^2 \\ &= 6 \times \frac{1}{2} \times 64 \times \frac{\sqrt{3}}{2} - 3\pi \times 16 \\ &= 96\sqrt{3} - 48\pi \\ &= 15.5 \text{ cm}^2\end{aligned}$$

6

Exploring randomness

Answers

Skills check

1 Write in ascending order: 85, 88, 91, 94, 95, 96, 97, 103, 103, 107, 110, 114

a Median is between 96 and 97 = 96.5 kg

b Mode = 103 kg (most frequent)

c Mean = total $\div$ number of observations
 $= 1183 \div 12$
 $= 98.6$ kg

d Range = $114 - 85 = 29$ kg

e Lower quartile = $\frac{13}{4}$ th observation
 $= 91 + \frac{1}{4} \times 3 = 91.75$ kg

Upper quartile = $\frac{3}{4} \times 13$ th observation
 $= 9\frac{3}{4}$ th observation
 $= 103 + \frac{3}{4} \times 4$
 $= 106$ kg

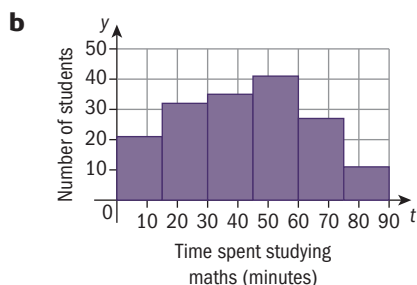
f IQR = $106 - 91.75 = 14.25$ kg

2 a $\binom{8}{3} = \frac{8 \times 7 \times 6}{3!} = 8 \times 7 = 56$

b Total number of ways – number of ways where all 3 have brown eyes
 $= \binom{20}{3} - \binom{12}{3} = 920$

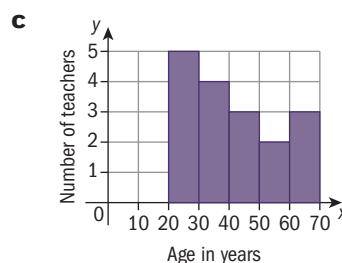
Exercise 6A

1 a Discrete (as they are asked for an answer in whole minutes)



2 a Continuous

b $5 + 4 + 3 + 2 + 3 = 17$



3 a Continuous

b

Mass (kg)	Number of chickens
$1 \leq w < 2$	8
$2 \leq w < 3$	24
$3 \leq w < 4$	50
$4 \leq w < 5$	14

c $8 + 24 + 50 + 14 = 96$

4 a Continuous

b

Time to get home (mins)	Number of students
$5 \leq t < 10$	1
$10 \leq t < 15$	2
$15 \leq t < 20$	4
$20 \leq t < 25$	4
$25 \leq t < 30$	2
$30 \leq t < 35$	2
$35 \leq t < 40$	1
$40 \leq t < 45$	1

c 5 mins

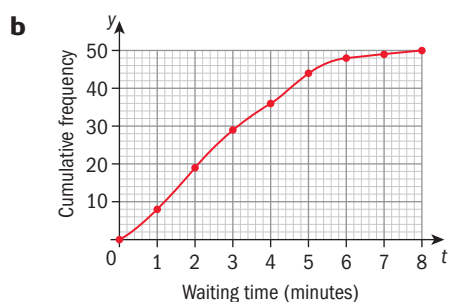
5 a First diagram = D
 Second diagram = A
 Third diagram = C

Exercise 6B

- 1 a 1 goal (highest frequency of 7)
b $170 \leq h < 180$ (highest frequency of 10)

2 a

t(minutes)	Frequency	CF
$0 \leq t < 1$	8	8
$1 \leq t < 2$	11	19
$2 \leq t < 3$	10	29
$3 \leq t < 4$	7	36
$4 \leq t < 5$	8	44
$5 \leq t < 6$	4	48
$6 \leq t < 7$	1	49
$7 \leq t < 8$	1	50
	50	



% waiting longer than 5 minutes
 $= \frac{6}{50} \times 100\% = 12\%$

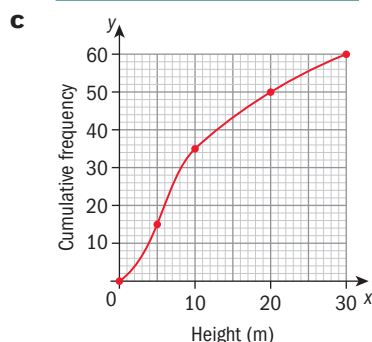
- c Estimates from table and graph:
 Mean ≈ 2.8 mins, Median ≈ 2.6 mins
 Modal interval is $1 \leq t < 2$ mins

3 a

Height (m)	Frequency
$0 \leq h < 5$	15
$5 \leq h < 10$	20
$10 \leq h < 20$	15
$20 \leq h < 30$	10
	60

b

Height (m)	CF
5	15
10	35
20	50
30	60



- c Number less than 18 m = 47
 $\therefore \% < 18 \text{ m} = \frac{47}{60} \times 100\% \approx 78\%$
 d Mean ≈ 11.0 m
 Median ≈ 9 m
 Modal class is $5 \leq h < 10$ m

- 4 a Mode = 3
 Median = 3

b Use: $\frac{30+a}{8} = \frac{a+3}{2}$

or $\frac{30+a}{8} = \frac{3+6}{2}$

Both give $a = 6$

This makes the set bimodal at 3 and 6.

- 5 a Series is $\ln a + \frac{1}{2} \ln a + \frac{1}{4} \ln a + \frac{1}{8} \ln a + \dots$
 This is a GP with common ratio $r = \frac{1}{2}$

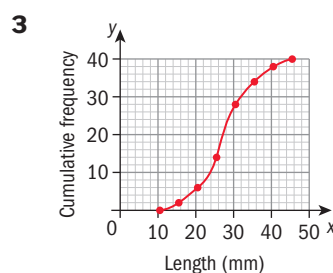
Since $|r| < 1$, this converges with sum
 $= \frac{A}{1-r} = \frac{\ln a}{1-\frac{1}{2}} = 2 \ln a$

b Mean = $\frac{\sum_{r=1}^n 2^{\frac{1}{r-1}} \ln a}{n} = \frac{A(1-r^n)}{(1-r)n}$
 $= \frac{\ln a \left(1 - \frac{1}{2^n}\right)}{\frac{1}{2^n} n} = \frac{2 \ln a \left(1 - \frac{1}{2^n}\right)}{n}$

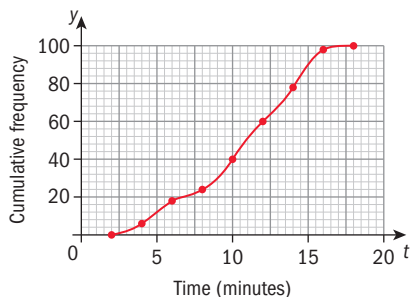
- c Need $\frac{2 \ln a}{n} \left(1 - \frac{1}{2^n}\right) < 0.01 \ln a$
 $\Rightarrow \left(1 - \frac{1}{2^n}\right) < 0.005 n$
 $\Rightarrow n = 200$

Exercise 6C

- 1 Arrange in order: 30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125
 a Range = $125 - 30 = 95$ cm
 b Median = $6\frac{1}{2}$ th reading = $\frac{65+70}{2} = 67.5$ cm
 c LQ = $\frac{13}{4}$ th reading = $55 + \frac{1}{4} \times 5 = 56.25$ cm
 d UQ = $9\frac{3}{4}$ th reading = $75 + \frac{3}{4} \times 35 = 101.25$ cm
 e IQR = $101.25 - 56.25$ cm
 2 a From graph, median = 75 cm
 b $77.5 - 72 = 5.5$ cm
 c 50% of the boxers have a reach with a maximum difference of 5.5 cm



4 a



i From graph, median = 11 min

ii IQR = 13.7 - 8.1 = 5.6 mins

b $24 + 36 + p = 92 \Rightarrow p = 32$

$24 + 36 + p + q = 100 \Rightarrow p + q = 40 \Rightarrow q = 8$

5 a From graph, number of students = 1100

b Lower quartile = $\frac{4200}{4}$ th = 1050th observation ≈ 39

Upper quartile = 3150th observation ≈ 64

Middle 50% lie between 39 and 64

$\Rightarrow a = 39, b = 64$

c Number getting more than 80 $\approx 4200 - 3900 = 300$

% awarded grade 7 $\approx \frac{300}{4200} \times 100\% = 7.1\%$

6 a 23 mins

b IQR = UQ - LQ = 31 - 16 = 15 mins

c 37 mins

Exercise 6D

1 $\frac{a+b+15}{6} = 3 \Rightarrow a+b = 3$
 $\frac{(a-3)^2 + (b-3)^2 + 1 + 0 + 4 + 4}{6} = \frac{7}{3}$

Solve to find that either $a = 2, b = 1$ or $a = 1, b = 2$.

Given that $a < b \therefore a = 1, b = 2$.

2 a Mean = $\frac{a-1+a+a+2+a+3}{4} = \frac{4a+4}{4} = a+1$

Variance = $\frac{(-2)^2 + (-1)^2 + 1^2 + 2^2}{4} + \frac{10}{4} = 2.5$

b Mean = $a+1+3 = a+4$

Variance = 2.5

3 a Mean = 9.4

Standard deviation = 1.41

b IQR = 10 - 9 = 1

4 $\frac{2+3+6+9+x+y}{6} = 6$

$\Rightarrow x+y = 36-20 \Rightarrow x+y = 16$ (1)

$\frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (9-6)^2 + (x-6)^2 + (y-6)^2}{6} = 10$

$\Rightarrow 16+9+9+(x-6)^2+(y-6)^2 = 60$

$\Rightarrow (x-6)^2 + (y-6)^2 = 60-34 = 26$ (2)

From (1), $y = 16 - x$, so

$(x-6)^2 + (10-x)^2 = 26$

$\therefore x^2 - 12x + 36 + 100 - 20x + x^2 = 26$

$\Rightarrow 2x^2 - 32x + 110 = 0$

$\Rightarrow x^2 - 16x + 55 = 0$

$\Rightarrow (x+5)(x-11) = 0$

x is positive, so $x = 11$ and $\therefore y = 5$ (from (1))

The last 2 score sums are 5 and 11

$\therefore$ score sums are 2, 3, 5, 6, 9, 11

Range = 11 - 2 = 9

IQR = 9.5 - 2.75 = 6.75

5 a Mean = $\frac{4k-2+k+k+1+2k+4+3k}{5} = \frac{11k+3}{5}$

b Variance

$= \frac{\sum x^2}{5} - \left(\frac{\sum x}{5} \right)^2$
 $= \frac{(4k-2)^2 + k^2 + (k+1)^2 + (2k+4)^2 + 9k^2}{5} - \left(\frac{11k+3}{5} \right)^2$
 $= \frac{34}{25}k^2 - \frac{56k^2}{25} + \frac{96}{25}$

c Mean = $\frac{11k+3}{5} - 2 = \frac{11k-7}{5}$

d The variance will be unchanged as the spread of the data about the mean is unaffected.

6 a Mean = $\frac{\sum_{r=1}^n (2r-1)a}{n} = \frac{a}{n} \left(2 \sum_{r=1}^n r - n \right)$
 $= \frac{a}{n} [n(n+1) - n] = \frac{a}{n} \times n^2 = an$

Exercise 6E

1 a $P(2, 4, 6, 8) = \frac{4}{2}$

b $P(3, 6) = \frac{1}{4}$

c $P(4, 8) = \frac{1}{4}$

d $P(1, 2, 3, 5, 6, 7) = \frac{3}{4}$

e $P(1, 2, 3) = \frac{3}{8}$

2 $\frac{30}{150} = \frac{1}{5}$

3 a $P(A) = \frac{1}{2}$

b $P(B) = \frac{4}{6} = \frac{2}{3}$

c $P(A \cup B) = \frac{10}{12} = \frac{5}{6}$

d $P(A \cap B) = \frac{4}{12} = \frac{1}{3}$

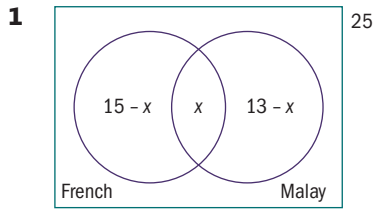
e $P(A' \cup B) = \frac{10}{12} = \frac{5}{6}$

5 a $P(A) = \frac{27}{36} = \frac{3}{4}$

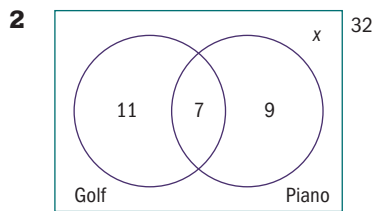
b $P(B) = \frac{18}{36} = \frac{1}{2}$

$$\begin{aligned} \text{c } P(A \cup B) &= \frac{27}{36} = \frac{3}{4} \\ \text{d } P(A \cap B) &= \frac{18}{36} = \frac{1}{2} \\ \text{e } P(A' \cup B') &= \frac{18}{36} = \frac{1}{2} \end{aligned}$$

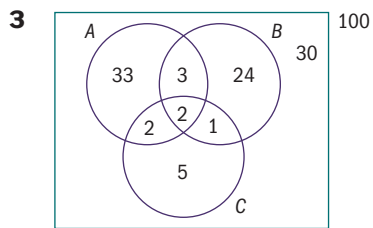
Exercise 6F



$$\begin{aligned} 15 - x + x + 13 - x + 5 &= 25 \\ \Rightarrow 28 - x &= 20 \Rightarrow x = 8 \\ \therefore P(\text{French and Malay}) &= \frac{8}{25} \end{aligned}$$



$$\begin{aligned} \text{a } P(\text{golf but not piano}) &= \frac{11}{32} \\ \text{b } P(\text{piano but not golf}) &= \frac{9}{32} \end{aligned}$$



$$\begin{aligned} \text{a } \frac{33}{100} &= 0.33 \\ \text{b } \frac{24}{100} &= 0.24 \\ \text{c } \frac{30}{100} &= 0.3 \end{aligned}$$

$$\begin{aligned} \text{4 a } P(X \cup Y) &= P(X) + P(Y) - P(X \cap Y) \\ &= \frac{1}{4} + \frac{1}{8} - \frac{1}{8} \\ &= \frac{1}{4} \end{aligned}$$

$$\text{b } P(X \cup Y)' = 1 - P(X \cup Y) = \frac{3}{4}$$

$$\begin{aligned} \text{5 a } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.2 + 0.5 - 0.1 = 0.6 \end{aligned}$$

$$\text{b } P(A \cup B)' = 1 - 0.6 = 0.4$$

$$\begin{aligned} \text{c } P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) \\ &= 0.8 + 0.5 - [P(B) - P(A \cap B)] \\ &= 0.5 + 0.5 - 0.5 + 0.1 = 0.6 \end{aligned}$$

Exercise 6G

$$\begin{aligned} \text{1 } P(\text{Sophie and Jerome selected}) &= \frac{\binom{8}{2}}{\binom{10}{4}} \\ &= \frac{2}{15} \end{aligned}$$

$$\text{2 a } P(\text{two lines}) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{8 \times 7} = \frac{3}{56}$$

$$\text{b } P(\text{two different pieces}) = \frac{2 \times \binom{3}{1} \times \binom{5}{1}}{\binom{8}{2}} = \frac{2 \times 3 \times 5}{8 \times 7} = \frac{15}{28}$$

$$\text{3 a } P(R, R, R) = \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} = \frac{7}{44}$$

$$\begin{aligned} \text{b } P(\text{not all same color}) &= 1 - P(R, R, R) - P(Y, Y, Y) \\ &= 1 - \frac{7}{44} - \left(\frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} \right) \\ &= \frac{35}{44} \end{aligned}$$

$$\text{4 a } P(\text{all orange}) = \frac{\binom{5}{3}}{\binom{15}{3}} = \frac{10}{455} = \frac{2}{91}$$

$$\begin{aligned} \text{b } P(\text{all different colors}) &= \frac{\binom{4}{1} \times \binom{5}{1} \times \binom{6}{1}}{\binom{15}{3}} = \frac{120}{455} = \frac{24}{91} \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{at least one green}) &= 1 - P(\text{no green}) \\ &= \frac{\binom{11}{2}}{\binom{15}{3}} = \frac{55}{455} = \frac{11}{91} \end{aligned}$$

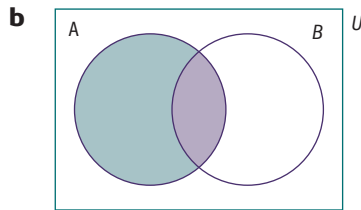
$$\begin{aligned} \text{5 a i } P(\text{Bob scores on 1st shoot}) &= P(\text{Bill misses}) \times P(\text{Bob hits}) \\ &= 0.7 \times 0.25 = 0.175 \\ \text{ii } P(\text{Bill scores on 3rd shoot}) &= P(\text{Bill misses, Bob misses, Bill misses, Bob misses, Bill hits}) \\ &= 0.7 \times 0.75 \times 0.7 \times 0.75 \times 0.3 \approx 0.0827 \\ \text{iii } P(\text{Bill scores on } n\text{th shoot}) &= P(\text{Bill misses, } n \text{ times, Bob misses, } n-1 \text{ times, Bob hits}) \\ &= (0.7)^n \times (0.75)^{n-1} \times 0.25 \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{Bill wins}) &= 0.3 + 0.7 \times 0.75 \times 0.3 + 0.7 \\ &\quad \times 0.75 \times 0.7 \times 0.75 \times 0.3 \\ &= 0.3 \left(1 + 0.525 + (0.525)^2 + \dots + 3 \right) \\ &= 0.3 \times \frac{1}{1-0.525} \\ \therefore p &= \frac{0.3}{1-0.525} \Rightarrow p - 0.525p = 0.3 \\ \Rightarrow p &= 0.3 + 0.525p \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{Bob wins}) &= 1 - P(\text{Bill wins}) \\ &= 1 - p \\ &= 1 - \frac{0.3}{1-0.525} \approx 0.368 \end{aligned}$$

Exercise 6H

$$\begin{aligned} 1 \text{ a } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.4 + 0.6 - 0.7 \\ &= 0.3 \end{aligned}$$



$$\begin{aligned} P(A \cap B') &= P(A) - P(A \cap B) \\ &= 0.4 - 0.3 = 0.1 \end{aligned}$$

$$\text{c } P(A' \cup B') = 1 - P(A \cap B) = 0.7$$

$$\begin{aligned} 2 \text{ } P(160 < h < 180) &= P(h < 180) - P(h < 160) \\ &= 0.75 - 0.2 = 0.55 \end{aligned}$$

$$\begin{aligned} 3 \text{ a } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.55 - 0.2 = 0.95 \end{aligned}$$

$$\begin{aligned} \text{b } P(A' \cap B) &= P(B) - P(A \cap B) \\ &= 0.55 - 0.2 = 0.35 \end{aligned}$$

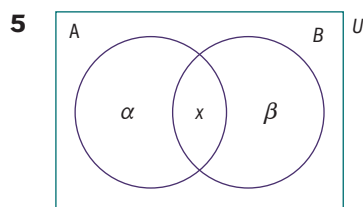
$$\begin{aligned} \text{c } P[(A \cup B) \setminus (A \cap B)] &= P(A' \cap B) \\ &\quad + P(B' \cap A) \end{aligned}$$

$$\begin{aligned} &= 0.35 + P(A) - P(A \cap B) \\ &= 0.35 + 0.6 - 0.2 = 0.75 \end{aligned}$$

$$4 \text{ a } P(A) = 0.5 + 0.2 = 0.7$$

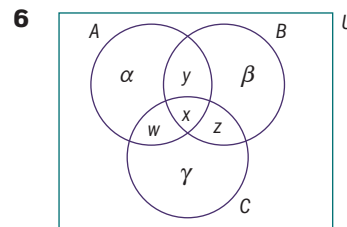
$$\begin{aligned} \text{b } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \therefore 0.85 &= 0.7 + P(B) - 0.2 \\ \therefore P(B) &= 0.35 \end{aligned}$$

$$\begin{aligned} \text{c } P(A' \cap B) &= P(B) - P(A \cap B) \\ &= 0.15 \end{aligned}$$

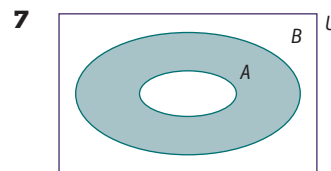


$$\begin{aligned} P(A) \times P(B) &= (\alpha + x)(\beta + x) \\ &= \alpha\beta + x(\alpha + \beta + x) \end{aligned}$$

$$\begin{aligned} \text{But } \alpha\beta > 0, \text{ so } P(A) \times P(B) &\geq x(\alpha + \beta + x) \\ &= P(A \cap B)P(A \cup B) \end{aligned}$$



$$\begin{aligned} P(A \cup B \cup C) &= \alpha + \beta + \gamma + x + y + z + w \\ &= (\alpha + x + y + w) + (\beta + x + y + z) \\ &\quad + (\gamma + w + x + z) - 2x - y - z - w \\ &= P(A) + P(B) + P(C) \\ &\quad - [(x + y) + (w + x) + (x + z) - x] \\ &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + x \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\quad - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$



$$\begin{aligned} P(B \setminus A) &= P(B \cap A') = P(B) + P(A') - P(B \cap A') \\ &= P(B) + P(A') - 1 = P(B) + 1 - P(A) - 1 \\ &= P(B) - P(A) \end{aligned}$$

Exercise 6I

$$1 \text{ a i } 0.21$$

$$\text{ii } 0.19 + 0.14 = 0.33$$

$$\text{b } 1200 \times 0.21 = 252$$

$$2 \text{ a } \frac{27}{100} = 0.27$$

b No. If it was fair we would expect around 16 or 17 occurrences of each number. The spinner appears to be biased towards 1.

$$\text{c } 0.15 \times 3000 = 450$$

$$3 \text{ a } P(5, 10) = \frac{34 + 68}{100} = \frac{102}{500} = \frac{51}{250}$$

$$\text{b } P(2, 4, 6, 8, 10, 12) = \frac{6 + 21 + 65 + 63 + 68 + 42}{500} = \frac{265}{500} = \frac{53}{100}$$

$$\text{c } P(2, 4, 6, 8, 10, 12, 5) = \frac{265 + 34}{500} = \frac{299}{500}$$

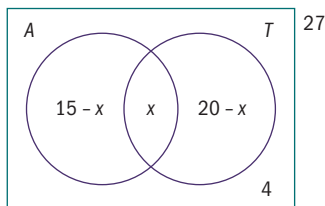
$$4 \text{ a } P(2, 3, 5, 7) = \frac{4}{10} = \frac{2}{5}$$

$$\text{b } P(2, 3, 5, 7, 4, 8) = \frac{6}{10} = \frac{3}{5}$$

$$\text{c } P(3, 6, 9, 4, 8) = \frac{5}{10} = \frac{1}{2}$$

Exercise 6J

1 a



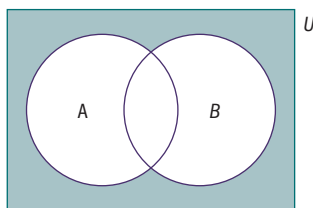
$$15 - x + x + 20 - x = 23 \quad \therefore 35 - x = 23 \quad \therefore x = 12$$

$$P(T \setminus A) = \frac{20 - x}{27} = \frac{8}{27}$$

$$b \quad P(A \cup T) = \frac{3 + 12 + 8}{27} = \frac{23}{27}$$

$$c \quad P(T|A) = \frac{P(T \cap A)}{P(A)} = \frac{\frac{12}{27}}{\frac{15}{27}} = \frac{12}{15} = \frac{4}{5}$$

2 a



$$P(A' \cap B') = 1 - P(A \cup B)$$

$$\Rightarrow P(A \cup B) = 1 - 0.35 = 0.65$$

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.25 + 0.6 - 0.65 \\ &= 0.2 \end{aligned}$$

$$b \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.6} = \frac{1}{3}$$

$$c \quad P(B'|A') = \frac{P(B' \cap A')}{P(A')} = \frac{0.35}{1 - 0.25} = \frac{0.35}{0.75} = \frac{7}{15}$$

$$\begin{aligned} 3 \quad P(\text{roller} | \text{skateboard}) &= \frac{P(\text{roller} \cap \text{skateboard})}{P(\text{skateboard})} \\ &= \frac{0.39}{0.48} = \frac{13}{16} \end{aligned}$$

$$\begin{aligned} 4 \quad a \quad P(\text{even} | \text{not mult. of 4}) &= \frac{P(\text{even} \cap \text{not mult. of 4})}{P(\text{not mult. of 4})} \\ &= \frac{P(2, 22)}{\frac{6}{8}} = \frac{\frac{8}{6}}{\frac{6}{8}} = \frac{1}{3} \end{aligned}$$

$$b \quad P(<15 | >5) = \frac{P(5 < x < 15)}{P(x < 15)} = \frac{\frac{2}{5}}{\frac{5}{8}} = \frac{2}{5}$$

$$c \quad P(<5 | <15) = \frac{P(x < 5 \text{ and } x < 15)}{P(x < 15)} = \frac{\frac{3}{5}}{\frac{5}{8}} = \frac{3}{5}$$

$$\begin{aligned} d \quad P(10 < x < 20 | 5 < x < 25) &= \frac{P(10 < x < 20 \text{ and } 5 < x < 25)}{P(5 < x < 25)} \\ &= \frac{\frac{2}{8}}{\frac{8}{4}} = \frac{1}{2} \end{aligned}$$

$$5 \quad P(\text{laptop} | \text{desktop}) = \frac{P(\text{laptop and desktop})}{P(\text{desktop})} = \frac{0.61}{0.95} = \frac{61}{95}$$

$$6 \quad P(\text{Spanish} | \text{Tech}) = \frac{P(\text{Spanish and Tech})}{P(\text{Tech})} = \frac{0.1}{0.6} = \frac{1}{6}$$

$$7 \quad a \quad P(U \text{ and } V) = P(U \cap V) = 0 \quad (\text{mutually exclusive})$$

$$b \quad P(U|V) = \frac{P(U \cap V)}{P(V)} = 0$$

$$c \quad P(U \text{ or } V) = P(U) + P(V) = 0.63$$

$$\begin{aligned} 8 \quad P(\text{passed 2} | \text{passed 1}) &= \frac{P(1 \text{ and } 2)}{P(1)} = \frac{0.35}{0.52} = 0.673 \\ \therefore 67.3\% \text{ of those who passed the first also passed the second.} \end{aligned}$$

$$9 \quad P(\text{white on 2nd} | \text{black on 1st})$$

$$= \frac{P(\text{1st black and 2nd white})}{P(\text{black on 1st})}$$

$$= \frac{0.34}{0.47} = \frac{34}{47}$$

$$10 \quad a \quad P(\text{male} \cap \text{left handed}) = \frac{5}{50} = \frac{1}{10}$$

$$b \quad P(\text{right handed}) = \frac{43}{50}$$

$$c \quad P(\text{right handed} | \text{female})$$

$$= \frac{P(\text{right handed and female})}{P(\text{female})} = \frac{\frac{11}{50}}{\frac{13}{50}} = \frac{11}{13}$$

$$\begin{aligned} 11 \quad P(\text{other is male} | \text{one is male}) &= \frac{P(\text{both male})}{P(\text{one is male})} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \\ &= \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{1}{3} \end{aligned}$$

Exercise 6K

$$1 \quad P(A \cap B) = 0.24 = 0.4 \times 0.6 = P(A) \times P(B)$$

$\therefore A$ and B independent

$$P(B \cap C) = 0.15 \neq P(B) \times P(C)$$

$\therefore B$ and C not independent

$$2 \quad P(A \cap B) = P(\text{Red Queen}) = \frac{2}{52} = \frac{1}{26}$$

$$P(A) \times P(B) = \frac{4}{52} \times \frac{1}{2} = \frac{1}{26}$$

$\therefore A$ and B independent

$$P(B \cap C) = P(\text{red face card}) = \frac{6}{52} = \frac{3}{26}$$

$$P(B) \times P(C) = \frac{1}{2} \times \frac{12}{52} = \frac{6}{52} = \frac{3}{26}$$

$\therefore B$ and C are independent

$$P(A \cap C) = P(\text{Queen and face card})$$

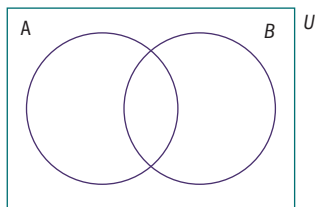
$$= P(\text{Queen}) = \frac{4}{52} = \frac{1}{13}$$

$$P(A) \times P(C) = \frac{4}{52} = \frac{12}{52}$$

$$= \frac{1}{13} \times \frac{3}{13} \neq \frac{1}{13}$$

$\therefore A$ and C are not independent

3



$$\begin{aligned} \text{a } P(A \cap B') &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B') \end{aligned}$$

$\therefore A$ and B' are not independent

$$\begin{aligned} \text{b } P(A' \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) \\ &= P(A')P(B) \end{aligned}$$

$\therefore A'$ and B are independent

$$\begin{aligned} \text{c } P(A' \cap B') &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A')P(B') \end{aligned}$$

$\therefore A'$ and B' are independent

$$\text{4 } P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{6} = \frac{1}{3} + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{6} = \frac{1}{3} + P(B) - \frac{1}{4} \Rightarrow P(B) = \frac{5}{6} - \frac{1}{3} + \frac{1}{4} = \frac{3}{4}$$

$$P(A) \times P(B) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4} = P(A \cap B)$$

$\therefore A$ and B independent

$$\begin{aligned} \text{5 a } P(A) \times P(B) &= P(A \cap B) \\ \Rightarrow P(B) &= \frac{P(A \cap B)}{P(A)} = \frac{0.18}{0.45} = \frac{2}{5} = 0.4 \end{aligned}$$

$$\begin{aligned} \text{b } P(A \cap B') &= P(A) - P(A \cap B) \\ &= 0.45 - 0.18 = 0.27 \end{aligned}$$

$$\begin{aligned} \text{c } P(A' \cap B') &= P(A') \times P(B') \\ &= 0.55 \times 0.6 = 0.33 \end{aligned}$$

$$\begin{aligned} \text{6 a } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 3a - \frac{5}{8} = P(A)P(B) \\ \Rightarrow 2a^2 - 3a + \frac{5}{8} &= 0 \Rightarrow 16a^2 - 24a + 5 = 0 \end{aligned}$$

$$\Rightarrow (4a - 5)(4a - 1) = 0 \Rightarrow a = \frac{5}{4} \text{ or } a = \frac{1}{4}$$

Since $P(A)$ and $P(B)$ must be ≤ 1 , $P(A) = \frac{1}{4}$,

$$P(B) = \frac{1}{2}$$

$$\text{7 a } P(T, 6) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

$$\text{b } P(H, \text{even}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\text{c } P(H, 3 \text{ or } 6) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\text{8 a } P(x, x, x, \text{even}) = 1 \times 1 \times 1 \times \frac{1}{2} = \frac{1}{2}$$

$$\text{b } P(x, x, x, 0 \text{ or } 5) = 1 \times 1 \times 1 \times \frac{2}{10} = \frac{1}{5}$$

$$\begin{aligned} \text{c } P(\text{divisible by } 4) &= P(\text{last 2 digits divisible by } 4) \\ &= P(x, x, 0, 0) + P(x, x, 0, 4) \\ &\quad + P(x, x, 0, 8) + \dots \\ &\quad + P(x, x, 9, 6) \\ &= \frac{25}{100} = \frac{1}{4} \end{aligned}$$

9 Since there is a large number of integers, can assume $P(\text{odd}) = P(\text{even}) = \frac{1}{2}$

Suppose we select n . Then $P(n \text{ even}) = \left(\frac{1}{2}\right)^n$

$$\therefore P(\text{at least one odd}) = 1 - \left(\frac{1}{2}\right)^n$$

$$\text{Need } 1 - \left(\frac{1}{2}\right)^n > 0.92$$

$$\Rightarrow \left(\frac{1}{2}\right)^n < 0.08$$

$$n \log 0.5 > \log 0.08$$

$$\Rightarrow n > 3.64 \therefore \text{need to select 4 integers}$$

$$\text{10 } P(\text{Julia fails to score a winner}) = 0.45$$

$$\therefore P(\text{Julia fails } n \text{ times in a row}) = (0.45)^n$$

$$\therefore P(\text{at least one winner in } n \text{ shots}) = 1 - (0.45)^n$$

$$1 - (0.45)^n > 0.999$$

$$\Rightarrow (0.45)^n < 0.001$$

$$\Rightarrow n > \frac{\log 0.001}{\log 0.45} \Rightarrow n > 8.65$$

$\therefore$ Julia needs to hit 9 shots

Exercise 6L

$$\text{1 } P(\text{Rain, not late}) = 0.2 \times 0.6 = 0.12$$

$$\text{2 } P(\text{Correct diagnosis}) = 0.85 \times 0.98 + 0.15 \times 0.12 = 0.851$$

$$\text{3 } P(1 \text{ Score out of } 2) = 0.75 \times 0.15 + 0.25 \times 0.8 = 0.3125$$

$$\begin{aligned} \text{4 a } P(B') &= \frac{1}{3} \times \frac{2}{5} + \frac{2}{3} \times \frac{1}{2} \\ &= \frac{2}{15} + \frac{1}{3} \\ &= \frac{7}{15} \end{aligned}$$

$$\begin{aligned} \text{b } P(A' \cup B') &= \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{2}{5} \\ &= \frac{2}{3} + \frac{2}{15} \\ &= \frac{12}{15} + \frac{2}{15} \end{aligned}$$

$$\begin{aligned} 5 \text{ a } P(\text{orange, orange, orange}) &= \frac{18}{30} \times \frac{17}{29} \times \frac{16}{28} \\ &= \frac{3}{5} \times \frac{17}{29} \times \frac{4}{7} \\ &= \frac{204}{1015} \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{at least one purple}) &= 1 - P(\text{all orange}) \\ &= 1 - \frac{204}{1015} = \frac{811}{1015} \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{more orange than purple}) &= P(o, o, o) + P(o, p, o) \\ &\quad + P(o, o, p) + P(p, o, o) \\ &= \frac{204}{1015} + \frac{18}{30} \times \frac{12}{29} \times \frac{17}{28} + \frac{18}{30} \times \frac{17}{29} \times \frac{12}{28} \\ &\quad + \frac{12}{30} \times \frac{18}{29} \times \frac{17}{28} \\ &= \frac{204}{1015} + \frac{459}{1015} = \frac{663}{1015} \end{aligned}$$

$$6 \text{ a } P(R, R, R) = \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} = \frac{2}{17}$$

$$\text{b } P(H, H, H) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} = \frac{11}{850}$$

$$\text{c } P(\text{all same suit}) = \frac{52}{52} \times \frac{12}{51} \times \frac{11}{50} = \frac{22}{425}$$

$$\text{d } P(\text{faces in same suit}) = \frac{12}{52} \times \frac{2}{51} \times \frac{1}{50} = \frac{1}{5525}$$

Exercise 6M

$$\begin{aligned} 1 \text{ a } P(\text{even}) &= \frac{1}{2} \times \frac{4}{9} + \frac{1}{2} \times \frac{3}{5} \\ &= \frac{2}{9} + \frac{3}{10} = \frac{47}{90} \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{first box} \mid \text{even}) &= \frac{P(\text{first box} \cap \text{even})}{P(\text{even})} \\ &= \frac{\frac{1}{2} \times \frac{4}{9}}{\frac{47}{90}} = \frac{20}{47} \end{aligned}$$

$$2 \text{ a } P(\text{defective}) = 0.6 \times 0.05 + 0.4 \times 0.02 = 0.038$$

$$\begin{aligned} \text{b } P(\text{first machine} \mid \text{defective}) &= \frac{P(f \cap d)}{P(d)} \\ &= \frac{0.6 \times 0.05}{0.038} = \frac{0.03}{0.038} \\ &= 0.789 \end{aligned}$$

$$3 \text{ a } P(V) = 0.6 \times 0.35 + 0.4 \times 0.75 = 0.51$$

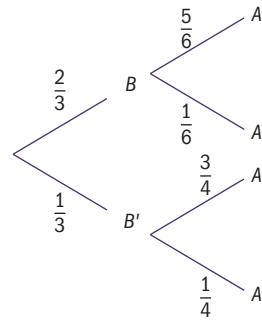
$$\text{b } P(V' \mid G) = 0.25$$

$$\begin{aligned} 4 \text{ a } P(\text{BB from 2nd box}) &= \frac{16}{30} \times \frac{13}{20} \times \frac{12}{19} + \frac{14}{30} \times \frac{12}{20} \times \frac{11}{19} \\ &= \frac{4344}{30 \times 20 \times 19} = 0.381 \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{1st box W} \mid \text{both from 2nd box W}) &= \frac{P(WWW)}{P(WW \text{ 2nd box})} \end{aligned}$$

$$\begin{aligned} &= \frac{\frac{14}{30} \times \frac{8}{20} \times \frac{7}{19}}{\frac{16}{30} \times \frac{7}{20} \times \frac{6}{19} + \frac{14}{30} \times \frac{8}{20} \times \frac{7}{19}} = \frac{784}{1456} = \frac{49}{91} = \frac{7}{13} \end{aligned}$$

5 a



$$\begin{aligned} \text{b i } P(A) &= \frac{2}{3} \times \frac{5}{6} + \frac{1}{3} \times \frac{3}{4} = \frac{5}{9} + \frac{1}{4} \\ &= \frac{20+9}{36} = \frac{29}{36} \end{aligned}$$

$$\text{ii } P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{2}{3} \times \frac{5}{6}}{\frac{29}{36}} = \frac{20}{29}$$

$$\text{iii } P(B' \mid A') = \frac{P(B' \cap A')}{P(A')} = \frac{\frac{1}{3} \times \frac{1}{4}}{\frac{2}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{4}} = \frac{3}{7}$$

$$6 \text{ a } P(6) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{6} = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

$$\begin{aligned} \text{b } P(\text{unbiased} \mid \text{not 6}) &= \frac{P(\text{unbiased and not 6})}{P(\text{not 6})} \\ &= \frac{\frac{1}{2} \times \frac{5}{6}}{\frac{1}{2} \times \frac{5}{6} + \frac{1}{2} \times \frac{1}{3}} = \frac{5}{7} \end{aligned}$$

$$7 \text{ a } P(\text{non-smoker}) = 0.18 \times 0.1 + 0.82 \times 0.8 = 0.674$$

$$\begin{aligned} \text{b } P(\text{lung problems} \mid \text{heavy smoker}) &= \frac{P(\text{lung problems and heavy smoker})}{P(\text{heavy smoker})} \\ &= \frac{0.18 \times 0.7}{0.18 \times 0.7 + 0.82 \times 0.05} = 0.754 \end{aligned}$$

$$\begin{aligned} 8 \text{ a } P(R) &= \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{3}{8} + \frac{1}{3} \times \frac{3}{5} \\ &= \frac{2}{9} + \frac{1}{8} + \frac{1}{5} = \frac{197}{360} \end{aligned}$$

$$\text{b } P(C \mid R) = \frac{P(C \cap R)}{P(R)} = \frac{\frac{1}{3} \times \frac{3}{5}}{\frac{197}{360}} = \frac{1}{5} \times \frac{360}{197} = \frac{72}{197}$$

$$\begin{aligned} 9 \text{ a } P(\text{on time}) &= 0.45 \times 0.95 + 0.2 \times 0.90 \\ &\quad + 0.35 \times 0.80 \\ &= 0.8875 \end{aligned}$$

$$\begin{aligned} \text{b } P(A \mid \text{on time}) &= \frac{P(A \text{ and on time})}{P(\text{on time})} \\ &= \frac{0.45 \times 0.95}{0.8875} \approx 0.482 \end{aligned}$$

$$\text{c } P(B \mid \text{late}) = \frac{P(B \text{ and late})}{P(\text{late})} = \frac{0.2 \times 0.1}{1 - 0.8875} \approx 0.178$$

$$\begin{aligned} 10 \text{ a } P(\text{Jar 2}, B) &= \frac{5}{15} \times \frac{4}{14} \times \frac{5}{11} + \frac{5}{15} \times \frac{10}{14} \times \frac{6}{11} \\ &\quad + \frac{10}{15} \times \frac{5}{14} \times \frac{6}{11} + \frac{10}{15} \times \frac{9}{14} \times \frac{7}{11} \\ &= \frac{1330}{15 \times 14 \times 11} = \frac{19}{33} \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{Jar 1} = PP \mid \text{Jar 2} = P) &= \frac{P(\text{PPP})}{P(\text{Jar 2} = P)} \\ &= \frac{\frac{5}{15} \times \frac{4}{14} \times \frac{6}{11}}{1 - P(B)} \\ &= \frac{5 \times 4 \times 6}{980} = \frac{6}{49} \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{Jar 1} = BB \mid \text{Jar 2} = P) \\ &= \frac{P(\text{BBP})}{P(\text{Jar 2} = P)} = \frac{\frac{10}{15} \times \frac{9}{14} \times \frac{4}{11}}{\frac{980}{15 \times 14 \times 11}} \\ &= \frac{18}{49} \end{aligned}$$

$$11 \text{ a } P(\text{male}) = 0.1 \times 0.6 + 0.65 \times 0.7 + 0.25 \times 0.3 = 0.59$$

$$\begin{aligned} \text{b } P(\text{management} \mid \text{male}) &= \frac{P(\text{male and management})}{P(\text{male})} \\ &= \frac{0.1 \times 0.6}{0.59} = 0.102 \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{marketing} \mid \text{female}) &= \frac{P(\text{female and marketing})}{P(\text{female})} \\ &= \frac{0.25 \times 0.7}{1 - 0.59} \approx 0.427 \end{aligned}$$

$$\begin{aligned} 12 \text{ } P(\text{second machine} \mid D') \\ &= \frac{P(\text{second machine and } D')}{P(D')} \\ &= \frac{0.35 \times 0.97}{0.5 \times 0.96 + 0.35 \times 0.97 + 0.15 \times 0.94} \\ &\approx 0.353 \end{aligned}$$

$$\begin{aligned} 13 \text{ } P(320, 320 \mid S = 160) \\ &= \frac{P(320, 320, 160)}{P(S = 160)} \\ &= \frac{\frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}}{\frac{12}{20} \times \frac{11}{19} \times \frac{10}{18} + \frac{12}{20} \times \frac{8}{19} \times \frac{11}{18} + \frac{8}{20} \times \frac{12}{19} \times \frac{11}{18} + \frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}} \\ &= \frac{672}{4104} = \frac{28}{171} \end{aligned}$$

$$14 \text{ } P(S) = \frac{4}{11} \times 0.9 + \frac{4}{11} \times 0.6 + \frac{3}{11} \times 0.2 = 0.6$$

$$\begin{aligned} 15 \text{ } P(\text{vowel}) &= \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} + \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} \\ &\quad + \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \\ &= \frac{90}{7 \times 6 \times 5} = \frac{3}{7} \end{aligned}$$



Review exercise

- 1 Mode = 6, so smallest set is 6, 6, x , y
 Median = 7, so set is 6, 6, 8, y
 Mean = 8, so $\frac{6+6+8+y}{4} = 8 \Rightarrow 20 + y = 32$
 $\therefore$ set is 6, 6, 8, 12

- 2 A and B independent

$$\Rightarrow P(A \mid B) = P(B) \Rightarrow P(B) = \frac{1}{3}$$

Let $P(A) = x$:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \therefore \frac{11}{12} &= x + \frac{1}{3} - x \times \frac{1}{3} \\ \Rightarrow \frac{2}{3}x + \frac{1}{3} &= \frac{11}{12} \\ \frac{2}{3}x &= \frac{7}{12} \Rightarrow x = \frac{7}{8} \end{aligned}$$

- 3 From graph:

- a Median = 68 kg
 b Middle 50% = 61 – 77 kg
 c There are 36 students

$$4 \text{ a } \binom{12}{3} = 220$$

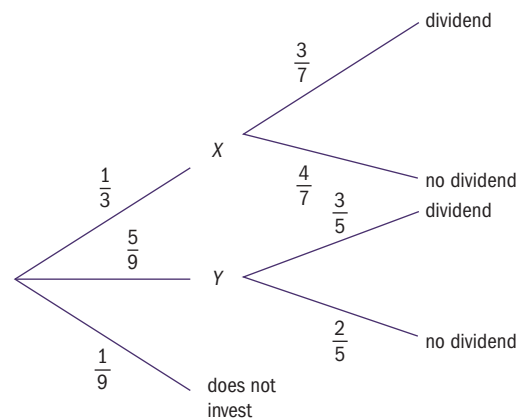
- b Probability = $1 - P(\text{both on committee})$

$$= 1 - \frac{\binom{10}{1}}{\binom{12}{2}} = 1 - \frac{10}{220} = \frac{11}{22}$$

- c $P(2 \text{ girls, 1 boy}) + P(3 \text{ girls})$

$$\begin{aligned} &= \frac{\binom{5}{2} \times \binom{7}{1}}{220} + \frac{\binom{5}{3}}{220} \\ &= \frac{80}{220} = \frac{4}{11} \end{aligned}$$

- 5 a



$$\begin{aligned} \text{b } P(\text{dividend}) &= \frac{1}{3} \times \frac{3}{7} + \frac{5}{7} \times \frac{3}{5} \\ &= \frac{1}{7} + \frac{1}{3} = \frac{10}{21} \end{aligned}$$

$$\begin{aligned} \text{c } P(Y \mid \text{dividend}) &= \frac{P(Y \text{ and dividend})}{P(\text{dividend})} = \frac{\frac{5}{9} \times \frac{3}{5}}{\frac{10}{21}} \\ &= \frac{7}{10} \end{aligned}$$

7 a $P(\text{prime}) = \frac{6}{36} = \frac{1}{6}$

b $P(\text{even}) = \frac{27}{36} = \frac{3}{4}$

c $P(\text{multiple of 3}) = \frac{20}{36} = \frac{5}{9}$

$$\begin{aligned} \mathbf{d} \quad P(\text{divisible by 6} \mid \text{even}) &= \frac{P(\text{divisible by 6 and even})}{P(\text{even})} \\ &= \frac{\frac{15}{36}}{\frac{3}{4}} = \frac{5}{9} \end{aligned}$$

8 a i $\text{mean} = \frac{\sum mi}{n} = \frac{540}{30} = 18$

ii Variance = $\frac{\sum m_i^2}{n} - \left(\frac{\sum m_i}{n}\right)^2$
 $= \frac{9990}{30} - 18^2 = 333 - 324 = 9$
 $\therefore \text{SD} = 3$

b No, as 95% of the students should get marks within 2 standard deviations of the mean, i.e. between 12 and 24, and 99.7% within 3 standard deviations of the mean, i.e. between 9 and 27.

9 a $\frac{3n+1}{2}$

b There are n number of the form $3k$ as k runs from 1 to n . But every other one is even, so number of odd numbers is $\frac{n+1}{2}$

$$\text{Hence } P(\text{divisible by } 3) = \frac{\frac{n+1}{2}}{\frac{3n+1}{2}} = \frac{n+1}{3n+1}$$

3 a $\frac{10 \times 10 \times 5}{10 \times 10 \times 10} = \frac{1}{2}$

b $P(\text{abc divisible by 7}) = \frac{142}{1000} = \frac{71}{500}$

c $P(\text{abc} = \text{perfect square}) = P(x^2, 1 \leq x \leq 31)$
 $= \frac{31}{1000}$

4 Probaility = $1 - 0.93 \times 0.93$
 $= 0.1351$

5 a Mean = 337.5 cm

b Standard deviation = 132.64 km

6

7

8 a Mean = 4.69

Standard deviation = 0.552

RBC	CF
3.4	0
3.8	7
4.2	22
4.6	58
5.0	80
5.4	107
5.8	120

Median = 60th observation = 4.64

c Number of children with
RBC $> 5.5 \approx 120 - 110 = 10$

$$\begin{aligned} \mathbf{9 \quad a} \quad P(\text{all stats books in first 6 places}) &= \frac{\frac{14!}{7!4!3!}}{\frac{20!}{6!7!4!3!}} \\ &= \frac{14!6!}{20!} = \frac{1}{38760} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(\text{all calculus books together}) &= \frac{\frac{14!}{6!4!3!}}{\frac{20!}{6!7!4!3!}} \\ &= \frac{14!7!}{20!} = \frac{7}{38760} \end{aligned}$$

10 a $P(\text{Keith wins bet i.e. } \mathcal{A} \text{ winning})$
 $= \frac{4}{11} \times 0.4 + \frac{3}{11} \times 0.55 + \frac{4}{11} \times 0.75$
 ≈ 0.568

$$\mathbf{b} \quad P(A \text{ played a higher rank} \mid A \text{ lost}) = \frac{\frac{4}{11} \times 0.6}{1 - 0.568} = 0.505$$



Review exercise

$$\mathbf{1} \quad \text{Mean height} = \frac{23 \times 168 + 17 \times 171 + 8 \times 163 + 20 \times 177}{68}$$
$$= 170.8\text{cm}$$

2 a $\frac{10!}{3!3!2!} = 50400$

b $P(S_{\text{-----}}) = \frac{9!}{2!3!2!} = \frac{15120}{50400} = \frac{3}{10}$

$$\begin{aligned} \mathbf{c} \quad & P(\text{-----consonant}) \\ &= 1 - \frac{9!}{50400} - \frac{9!}{50400} \\ &= 1 - \frac{5040}{50400} - \frac{10080}{50400} \\ &= 1 - \frac{15120}{50400} = \frac{7}{10} \end{aligned}$$

7

The evolution of calculus

Answers

Skills check

1 a $y = x \ln x \quad \frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$

b $y = \frac{e^{2x-3}}{\sqrt{2-x}}$

$$\frac{dy}{dx} = \frac{\left((2-x)^{\frac{1}{2}} 2e^{2x-3} + e^{2x-3} \frac{1}{2}(2-x)^{-\frac{1}{2}} \right)}{(2-x)}$$

$$= \frac{4(2-x)e^{2x-3} + e^{2x-3}}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{e^{2x-3}(9-4x)}{2(2-x)^{\frac{3}{2}}}$$

c $y = x^4 - \frac{1}{x^4}$

$$\frac{dy}{dx} = 4x^3 + 4x^{-5} = 4x^3 + \frac{4}{x^5}$$

2 a $y = 3x - 2 \quad y^2 = x^2 - 2x + 4$

$$x^2 - 2x + 4 = 3x - 2$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$x = 2 \text{ or } 3 \quad (2, 4) \text{ or } (3, 7)$$

b $y = 1 - x \quad y = \sqrt{2x+1}$

$$1 - x = \sqrt{2x+1} \quad (1)$$

$$(1-x)^2 = 2x+1$$

$$1 - 2x + x^2 = 2x+1$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \text{ or } 4$$

Check in (1)

if $x = 0$, LHS = 1 RHS = 1 ✓

if $x = 4$, LHS = -3 RHS = 3 ✗

$$\therefore x = 0 \quad (0, 1)$$

c $y = \frac{6}{x} + 3x \quad y = x^3 - 5x$

$$\frac{6}{x} + 3x = x^3 - 5x$$

$$6 + 3x^2 = x^4 - 5x^2$$

$$x^4 - 8x^2 - 6 = 0$$

$$x = -2.948 \text{ or } 2.948$$

$$(-2.95, -10.9) \text{ or } (2.95, 10.9)$$

3 $s(t) = 3t^4 - t^3 + t$

$$v(t) = s'(t) = 12t^3 - 3t^2 + 1$$

$$a(t) = v'(t) = 36t^2 - 6t$$

Exercise 7A

1 $\int -2x \, dx = -x^2 + c$

2 $\int 3x^8 \, dx = \frac{x^9}{3} + c$

3 $\int -5x^4 \, dx = -x^5 + c$

4 $\int \frac{1}{x^5} \, dx = \int x^{-5} \, dx = \frac{-x^{-4}}{4} + c = \frac{-1}{4x^4} + c$

5 $\int \sqrt{x^3} \, dx = \int x^{\frac{3}{2}} \, dx = \frac{2}{5} x^{\frac{5}{2}} + c$

6 $\int \frac{1}{\sqrt{x^3}} \, dx = \int \frac{1}{x^{\frac{3}{2}}} \, dx = \int x^{-\frac{3}{2}} \, dx = -2x^{-\frac{1}{2}} + c = \frac{-2}{\sqrt{x}} + c$

7 $\int \frac{2x}{\sqrt{x}} \, dx = \int 2x^{\frac{1}{2}} \, dx = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{3} x^{\frac{3}{2}} + c$

8 $\int \frac{-\sqrt{x^5}}{7x^3} \, dx = \int -\frac{1}{7} x^{\frac{-7}{4}} \, dx = -\frac{1}{7} x^{\frac{-3}{4}} \left(\frac{-4}{3} \right) + c$

$$= \frac{4}{21} x^{\frac{-3}{4}} + c$$

Exercise 7B

1 a $\int \left(5x^2 - \frac{1}{5x^2} \right) dx = \int \left(5x^2 - \frac{1}{5} x^{-2} \right) dx$

$$= \frac{5x^3}{3} + \frac{1}{5x} + c$$

b $\int (x+3)(2x-1) dx = \int (2x^2 + 5x - 3) dx$

$$= \frac{2x^3}{3} + \frac{5x^2}{2} - 3x + c$$

c $\int \frac{x^2-1}{x^4} \, dx = \int (x^{-2} - x^{-4}) \, dx$

$$= -x^{-1} + \frac{x^{-3}}{3} + c = -\frac{1}{x} + \frac{1}{3x^3} + c$$

d $\int \left(x + \frac{1}{x} \right)^2 dx = \int (x^2 + 2 + x^{-2}) dx$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

e $\int \frac{(x+3)(x-4)}{x^5} \, dx = \int (x^{-3} - x^{-4} - 12x^{-5}) dx$

$$= \frac{-x^{-2}}{2} + \frac{x^{-3}}{3} + \frac{12x^{-4}}{4} + c$$

$$= -\frac{1}{2x^2} + \frac{1}{3x^3} + \frac{3}{x^4} + c$$

$$\begin{aligned} \mathbf{f} \quad \int \left(\sqrt{x} - \frac{5}{\sqrt[3]{x}} \right) dx &= \int \left(x^{\frac{1}{2}} - 5x^{\frac{-1}{3}} \right) dx \\ &= \frac{2}{3} x^{\frac{3}{2}} - 5x^{\frac{2}{3}} \left(\frac{3}{2} \right) + c \\ &= \frac{2}{3} x^{\frac{3}{2}} - \frac{15}{2} x^{\frac{2}{3}} + c \end{aligned}$$

$$\mathbf{2} \quad \frac{dy}{dx} = (3x^2 - 4) \quad (2, -1)$$

$$y = x^3 - 4x + c$$

$$-1 = 8 - 8 + c \quad \therefore c = -1$$

$$y = x^3 - 4x - 1$$

$$\mathbf{3} \quad f'(t) = t + 3 - \frac{1}{t^2} \quad \left(1, \frac{-1}{2}\right)$$

$$f(t) = \frac{t^2}{2} + 3t + \frac{1}{t} + c$$

$$-\frac{1}{2} = \frac{1}{2} + 3 + 1 + c \quad \therefore c = -5$$

$$f(t) = \frac{t^2}{2} + 3t + \frac{1}{t} - 5$$

$$\mathbf{4} \quad \frac{dy}{dx} = (2x + 3)^3 = 8x^3 + 3(2x)^2(3) + 3(2x)3^2 + 3^3$$

$$= 8x^3 + 36x^2 + 54x + 27$$

$$y = 2x^4 + 12x^3 + 27x^2 + 27x + c$$

$$z = 2 - 12 + 27 - 27 + c \quad \therefore c = 12$$

$$y = 2x^4 + 12x^3 + 27x^2 + 27x + 12 = \frac{(2x+3)^4 + 15}{8}$$

$$\mathbf{5} \quad \frac{dA}{dx} = (2x + 1)(x^2 - 1) = 2x^3 + x^2 - 2x - 1$$

$$A = \frac{x^4}{2} + \frac{x^3}{3} - x^2 - x + c$$

$$0 = \frac{1}{2} + \frac{1}{3} - 1 - 1 + c \quad \therefore c = \frac{7}{6}$$

$$A = \frac{x^4}{2} + \frac{x^3}{3} - x^2 - x + \frac{7}{6}$$

$$\mathbf{6} \quad \frac{ds}{dt} = 3t - \frac{8}{t^2}$$

$$s = \frac{3t^2}{2} + \frac{8}{t} + c$$

$$1.5 = 1.5 + 8 + c \quad \therefore c = -8$$

$$s = \frac{3t^2}{2} + \frac{8}{t} - 8$$

$$\mathbf{7} \quad \frac{d^2y}{dx^2} = 6x - 1 \quad \frac{dy}{dx} = 3x^2 - x + c$$

$$4 = 12 - 2 + c \quad \therefore c = -6$$

$$\frac{dy}{dx} = 3x^2 - x - 6$$

$$y = x^3 - \frac{x^2}{2} - 6x + c$$

$$0 = 8 - 2 - 12 + c \quad \therefore c = 6$$

$$y = x^3 - \frac{x^2}{2} - 6x + 6$$

$$\mathbf{8} \quad a(t) = 6t + 1$$

$$v(t) = 3t^2 + t + c$$

$$2 = c \quad \therefore v(t) = 3t^2 + t + 2$$

$$s(t) = t^3 + \frac{t^2}{2} + 2t + c$$

$$1 = c \quad \therefore s(t) = t^3 + \frac{t^2}{2} + 2t + 1$$

Exercise 7C

$$\mathbf{1} \quad \int (3x - 1)^7 dx = \frac{(3x-1)^8}{24} + c$$

$$\begin{aligned} \mathbf{2} \quad \int -2\sqrt{2x+1} dx &= \int -2(2x+1)^{\frac{1}{2}} dx \\ &= \frac{-2(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{-2(2x+1)^{\frac{3}{2}}}{3} + c \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \int \frac{1}{(4x-1)^5} dx &= \int (4x-1)^{-5} dx \\ &= \frac{(4x-1)^{-4}}{4(-4)} + c = \frac{-1}{16(4x-1)^4} + c \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \int \frac{2}{\sqrt[4]{3-x}} dx &= \int 2(3-x)^{-\frac{1}{4}} dx \\ &= \frac{2(3-x)^{\frac{3}{4}}}{-\frac{3}{4}} + c = \frac{-8(3-x)^{\frac{3}{4}}}{3} + c \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \int \left(\frac{2}{(2-5x)^{\frac{1}{3}}} + \sqrt[3]{1-x} \right) dx &= \int \left(2(2-5x)^{-\frac{1}{3}} + (1-x)^{\frac{1}{3}} \right) dx \\ &= \frac{2(2-5x)^{\frac{2}{3}}}{-\frac{5}{3}} + \frac{(1-x)^{\frac{4}{3}}}{\frac{4}{3}} + c \\ &= \frac{-3(2-5x)^{\frac{2}{3}}}{5} - \frac{3(1-x)^{\frac{4}{3}}}{4} + c \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \int \left(4\sqrt{2-3x} - 6(3x+2)^{\frac{2}{3}} \right) dx &= \int \left(4(2-3x)^{\frac{1}{2}} - 6(3x+2)^{\frac{2}{3}} \right) dx \\ &= \frac{4(2-3x)^{\frac{3}{2}}}{-\frac{3}{2}} - \frac{6(3x+2)^{\frac{5}{3}}}{\frac{5}{3}} + c \\ &= \frac{-8(2-3x)^{\frac{3}{2}}}{9} - \frac{6(3x+2)^{\frac{5}{3}}}{5} + c \end{aligned}$$

Exercise 7D

$$\mathbf{1} \quad \int -5e^{-2x} dx = \frac{5e^{-2x}}{2} + c$$

$$\begin{aligned} \mathbf{2} \quad \int \frac{1}{e^{3x+2}} dx &= \int e^{-3x-2} dx \\ &= \frac{-1}{3} e^{-3x-2} + c \end{aligned}$$

$$\begin{aligned} 3 \quad \int \left(\sqrt[3]{e^x} - \frac{2}{e\sqrt{e^{2x}}} \right) dx &= \int (e^{\frac{x}{3}} - 2e^{-x-1}) dx \\ &= 3e^{\frac{x}{3}} + 2e^{-x-1} + c \end{aligned}$$

$$4 \quad \int 3^x dx = \frac{3^x}{\ln 3} + c$$

$$5 \quad \int \frac{1}{3^{2x}} dx = \int 3^{-2x} dx = -\frac{3^{-2x}}{2\ln 3} + c$$

$$6 \quad \int 4^{1-x} dx = -\frac{4^{1-x}}{\ln 4} + c$$

$$\begin{aligned} 7 \quad \int m^{ax+b} dx \quad &\text{Let } u = ax + b \\ &\frac{du}{dx} = a \quad \therefore dx = \frac{du}{a} \\ \int m^{ax+b} dx &= \int m^u \frac{du}{a} = \frac{1}{a} \int m^u du \\ &= \frac{1}{a} \frac{m^u}{\ln(m)} + c \\ &= \frac{1}{a \ln(m)} m^{ax+b} + c \end{aligned}$$

Exercise 7E

$$1 \quad \int \frac{1}{3x} dx = \frac{1}{3} \ln|x| + c$$

$$2 \quad \int -\frac{6}{x} dx = -6 \ln|x| + c$$

$$3 \quad \int \frac{1}{2-3x} dx = -\frac{1}{3} \ln|2-3x| + c$$

$$4 \quad \int \frac{5}{3-5x} dx = -\ln|3-5x| + c$$

$$5 \quad \int -2(4+3x)^{-1} dx = -\frac{2}{3} \ln|4+3x| + c$$

Exercise 7F

$$\begin{aligned} 1 \quad \int_1^3 \left(3x + \frac{1}{x^2} \right) dx &= \left[\frac{3x^2}{2} - \frac{1}{x} \right]_1^3 = \frac{27}{2} - \frac{1}{3} - \frac{3}{2} + 1 \\ &= 12 - \frac{1}{3} + 1 \\ &= \frac{38}{3} \end{aligned}$$

$$\begin{aligned} 2 \quad \int_0^2 3\sqrt{4x+1} dx &= 3 \left[\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 \\ &= \frac{1}{2} \left[(4x+1)^{\frac{3}{2}} \right]_0^2 \\ &= \frac{1}{2} (27 - 1) = 13 \end{aligned}$$

$$\begin{aligned} 3 \quad \int_{-1}^2 -2e^{1-3x} dx &= \frac{2}{3} \left[e^{1-3x} \right]_{-1}^2 \\ &= \frac{2}{3} (e^{-5} - e^4) = \frac{2(1-e^9)}{3e^5} \end{aligned}$$

$$\begin{aligned} 4 \quad \int_1^3 3.2^{x+1} dx &= \frac{3}{\ln 2} \left[2^{x+1} \right]_1^3 \\ &= \frac{3}{\ln 2} (16 - 4) = \frac{36}{\ln 2} \end{aligned}$$

$$\begin{aligned} 5 \quad \int_{-2}^0 2(1-3x)^5 dx &= \frac{2}{-3(6)} \left[(1-3x)^6 \right]_{-2}^0 \\ &= -\frac{1}{9} [1 - 7^6] \\ &= 13072 \end{aligned}$$

$$\begin{aligned} 6 \quad \int_1^4 \frac{1-\sqrt{x}}{\sqrt{x}} dx &= \int_1^4 (x^{-\frac{1}{2}} - 1) dx \\ &= \left[2x^{\frac{1}{2}} - x \right]_1^4 \\ &= (4 - 4) - (2 - 1) = -1 \end{aligned}$$

Exercise 7G

$$\begin{aligned} 1 \quad \int_{-1}^0 (2r-1)^4 dr &= \left[\frac{(2r-1)^5}{10} \right]_{-1}^0 \\ &= \frac{1}{10} ((-1) - (-3)^5) = \frac{242}{10} = \frac{121}{5} \end{aligned}$$

$$2 \quad \text{Not possible, } s \neq 0$$

$$3 \quad \text{Not possible, } x \neq \pm 1, 1 \in [0, 2]$$

$$\begin{aligned} 4 \quad \int_0^1 \frac{dx}{(2x+1)^3} &= \int_0^1 (2x+1)^{-3} dx \\ &= -\frac{1}{4} \left[(2x+1)^{-2} \right]_0^1 = -\frac{1}{4} \left[\frac{1}{(2x+1)^2} \right]_0^1 \\ &= -\frac{1}{4} \left(\frac{1}{9} - 1 \right) = \frac{2}{9} \end{aligned}$$

$$5 \quad \text{Not possible, } x \neq -1$$

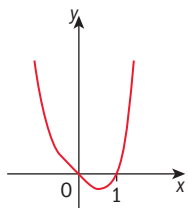
$$\begin{aligned} 6 \quad \int_0^1 \left(\frac{3}{3x+4} - \frac{2}{x+1} \right) dx &= [\ln|3x+4| - 2\ln|x+1|]_0^1 \\ &= (\ln 7 - 2\ln 2) - (\ln 4 - 2\ln 1) \\ &= \ln 7 - \ln 4 - \ln 4 \\ &= \ln \frac{7}{16} \end{aligned}$$

$$\begin{aligned} 7 \quad \int_{-1}^1 \frac{e^x + 4}{e^x} dx &= \int_{-1}^1 (1 + 4e^{-x}) dx \\ &= \left[x - 4e^{-x} \right]_{-1}^1 \\ &= (1 - 4e^{-1}) - (-1 - 4e) \\ &= 2 - \frac{4}{e} + 4e \end{aligned}$$

$$\begin{aligned} 8 \quad \int_0^2 10^x dx &= \frac{1}{\ln 10} \left[10^x \right]_0^2 \\ &= \frac{1}{\ln 10} (100 - 1) \\ &= \frac{99}{\ln 10} \end{aligned}$$

Exercise 7H

1 $y = x^4 - x = x(x^3 - 1)$



$$\int_{-1}^0 (x^4 - x) dx = \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_{-1}^0$$

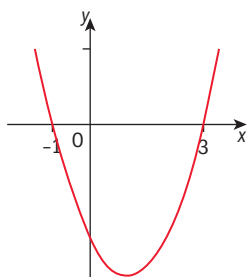
$$= 0 - \left(\frac{-1}{5} - \frac{1}{2} \right) = \frac{7}{10}$$

$$\int_0^1 (x^4 - x) dx = \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}$$

$$\therefore \text{area} = \frac{7}{10} + \frac{3}{10} = 1 \text{ sq. unit}$$

2 $y = x^2 - 2x - 3 = (x - 3)(x + 1)$



$$A = \int_{-1}^3 (x^2 - 2x - 3) dx$$

$$= \left[\frac{x^3}{3} - x^2 - 3x \right]_{-1}^3$$

$$= \left(\frac{27}{3} - 9 - 9 \right) - \left(-\frac{1}{3} - 1 + 3 \right)$$

$$= \frac{32}{3} \text{ sq. units}$$

3 $\int_{-1}^1 (x^2 - 2x - 3) dx = \left[\frac{x^3}{3} - x^2 - 3x \right]_{-1}^1$

$$= \left(\frac{1}{3} - 1 - 3 \right) - \left(-\frac{1}{3} - 1 + 3 \right) = -\frac{16}{3}$$

$$\therefore \text{required area} = \frac{32}{3} + \frac{16}{3} = 16 \text{ sq. units}$$

4 $\int_{\ln 3}^3 (e^x - 3) dx = [e^x - 3x]_{\ln 3}^3$

$$= e^3 - 9 - 3 + 3 \ln 3$$

$$= e^3 + 3 \ln 3 - 12$$

$$\int_0^{\ln 3} (e^x - 3) dx = [e^x - 3x]_0^{\ln 3}$$

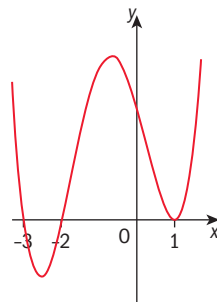
$$= 3 - 3 \ln 3 - 1$$

$$= -(3 \ln 3 - 2)$$

$$\therefore \text{area} = e^3 + 3 \ln 3 - 12 + 3 \ln 3 - 2$$

$$= e^3 + 6 \ln 3 - 14$$

5 $y = x^4 + 3x^3 - 3x^2 - 7x + 6$



$$\int_{-3}^{-2} (x^4 + 3x^3 - 3x^2 - 7x + 6) dx$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4} - x^3 - \frac{7x^2}{2} + 6x \right]_{-3}^{-2}$$

$$= \left(-\frac{32}{5} + 12 + 8 - 14 - 12 \right)$$

$$- \left(-\frac{243}{5} + \frac{243}{4} + 27 - \frac{63}{2} - 18 \right)$$

$$= -12.4 - (-10.35) = -2.05$$

$$\int_{-2}^1 (x^4 + 3x^3 - 3x^2 - 7x + 6) dx$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4} - x^3 - \frac{7x^2}{2} + 6x \right]_{-2}^1$$

$$= \left(\frac{1}{5} + \frac{3}{4} - 1 - \frac{7}{2} + 6 \right) - (-12.4) = 14.85$$

$$\text{area} = 2.05 + 14.85 = 16.9 \text{ sq. units}$$

6 $y = \sqrt{4 - x} \quad x = 0, \quad x = 4$

$$A = \int_0^4 (4 - x)^{\frac{1}{2}} dx = \left[-\frac{2}{3} (4 - x)^{\frac{3}{2}} \right]_0^4$$

$$= -\frac{2}{3} (0 - 4^{\frac{3}{2}}) = \frac{16}{3} \text{ sq. units}$$

7 $y = \frac{1}{x^2} + 1 \quad x = \frac{1}{2}, \quad x = 5$

$$A = \int_{\frac{1}{2}}^5 (x^{-2} + 1) dx = \left[-\frac{1}{x} + x \right]_{\frac{1}{2}}^5$$

$$= \left(-\frac{1}{5} + 5 \right) - \left(-2 + \frac{1}{2} \right)$$

$$= 6.3 \text{ sq. units}$$

8 $y = 2^x \quad x = 1, \quad x = 2$

$$A = \int_1^2 2^x dx = \frac{1}{\ln 2} [2^x]_1^2$$

$$= \frac{1}{\ln 2} (4 - 2) = \frac{2}{\ln 2} \text{ sq. units}$$

9 $y = 2e^{-x+1} - 1 \quad x = 0, \quad x = 3$

$$A = \int_0^3 |2e^{-x+1} - 1| dx = 3.32 \text{ sq. units}$$

10 $y = \frac{1}{x+2} \quad x = -1, \quad x = 2$

$$A = \int_{-1}^2 \frac{1}{x+2} dx = [\ln |x+2|]_{-1}^2$$

$$= \ln 4 - \ln 1 = \ln 4 = 2 \ln 2 \text{ sq. units}$$

$$11 \quad y = \frac{2}{3-4x} \quad x=1, x=3$$

$$\int_1^3 \frac{2}{3-4x} dx = \frac{2}{-4} [\ln|3-4x|]_1^3$$

$$= -\frac{1}{2}(\ln 9 - \ln 1) = -\frac{1}{2}\ln 9 = -\ln 3$$

$$\therefore \text{area} = \ln 3 \text{ sq. units}$$

$$12 \quad y = -x^3 + 6x^2 + x - 30 \quad x\text{-intercepts: } -2, 3, 5$$

$$\int_{-2}^3 (-x^3 + 6x^2 + x - 30) dx$$

$$= \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x \right]_{-2}^3$$

$$= \left(\frac{-81}{4} + 54 + \frac{9}{2} - 90 \right) - (-4 - 16 + 2 + 60)$$

$$= -51.75 - 42$$

$$= -93.75$$

$$\int_3^5 (-x^3 + 6x^2 + x - 30) dx = \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x \right]_3^5$$

$$= \left(\frac{-625}{4} + 250 + \frac{25}{2} - 150 \right) - (-51.75)$$

$$= -43.75 + 51.75 = 8$$

$$\therefore \text{area} = 93.75 + 8 = 101.75 \text{ sq. units}$$

$$13 \quad y = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}$$

$$\text{Area} = \int_0^1 x^2 dx + \int_1^2 (2-x) dx$$

$$= \left[\frac{x^3}{3} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2$$

$$= \frac{1}{3} + (4-2) - \left(2 - \frac{1}{2} \right) = \frac{5}{6} \text{ sq. units}$$

$$14 \quad y = \begin{cases} \sqrt{x} & 0 \leq x \leq 1 \\ x^2 & 1 \leq x \leq 2 \end{cases}$$

$$\text{Area} = \int_0^1 x^{\frac{1}{2}} dx + \int_1^2 x^2 dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^1 + \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{1}{3}$$

$$= 3 \text{ sq. units}$$

Exercise 7I

$$1 \quad y = x^2 + 1 \quad y=1, y=10$$

$$x = \sqrt{y-1} \quad A = \int_1^{10} (y-1)^{\frac{1}{2}} dy = \left[\frac{2}{3} (y-1)^{\frac{3}{2}} \right]_1^{10} = \frac{2}{3} (9)^{\frac{3}{2}} = 18 \text{ sq. units}$$

$$2 \quad y = \sqrt{x} \quad y=0, y=4$$

$$x = y^2$$

$$A = \int_0^4 y^2 dy = \left[\frac{y^3}{3} \right]_0^4 = \frac{64}{3} \text{ sq. units}$$

$$3 \quad y = \sqrt{4-x} \quad y=0, y=2$$

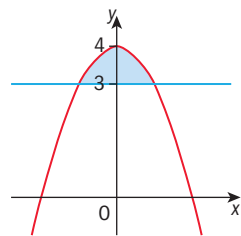
$$y^2 = 4-x$$

$$x = 4-y^2$$

$$A = \int_0^2 (4-y^2) dy = \left[4y - \frac{y^3}{3} \right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3} \text{ sq. units}$$

$$4 \quad y = 4-x^2 \quad y=3, y=4$$



$$x^2 = 4-y$$

$$x = (4-y)^{\frac{1}{2}}$$

$$A = 2 \int_3^4 (4-y)^{\frac{1}{2}} dy = 2 \left[\frac{-2}{3} (4-y)^{\frac{3}{2}} \right]_3^4$$

$$= \frac{-4}{3} (0-1) = \frac{4}{3} \text{ sq. units}$$

$$5 \quad y = \frac{1}{\sqrt{-x+4}} \quad y = \frac{1}{2} \quad y=2$$

$$-x+4 = \frac{1}{y^2} \quad x = 4 - \frac{1}{y^2}$$

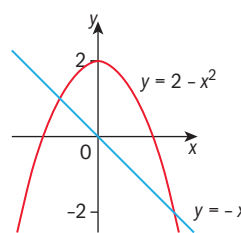
$$A = \int_{\frac{1}{2}}^2 \left(4 - \frac{1}{y^2} \right) dy = \left[4y + \frac{1}{y} \right]_{\frac{1}{2}}^2$$

$$= \left(8 + \frac{1}{2} \right) - \left(2 + 2 \right)$$

$$= 4 \frac{1}{2} \text{ sq. units}$$

Exercise 7J

1



$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } -1$$

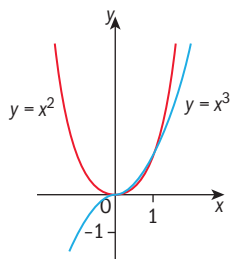
$$\text{Area} = \int_{-1}^2 (2 - x^2 + x) dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2$$

$$= \left(4 - \frac{8}{3} + 2 \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right)$$

$$= \frac{9}{2} \text{ sq. units}$$

2



$$x^3 = x^2$$

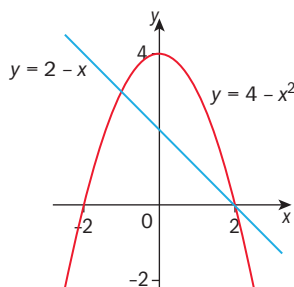
$$x^2(x-1) = 0$$

$$x = 0 \text{ or } 1$$

$$A = \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \text{ sq. units}$$

3



$$4 - x^2 = 2 - x$$

$$x^2 - x - 2 = 0$$

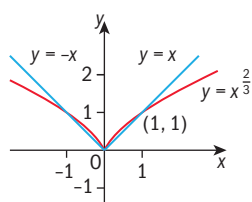
$$(x-2)(x+1) = 0$$

$$x = -1 \text{ or } 2$$

$$\text{Area} = \int_{-1}^2 (4 - x^2 - (2 - x)) dx = \int_{-1}^2 (2 - x^2 + x) dx$$

$$= \frac{9}{2} \text{ sq. units (see qn. 1)}$$

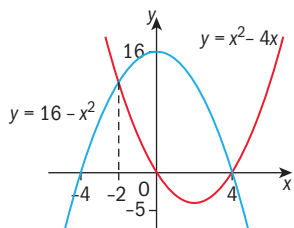
4



$$\text{Area} = 2 \int_0^1 (x^{2/3} - x) dx = 2 \left[\frac{3}{5} x^{5/3} - \frac{x^2}{2} \right]_0^1$$

$$= 2 \left(\frac{3}{5} - \frac{1}{2} \right) = \frac{1}{5}$$

5



$$16 - x^2 = x^2 - 4x$$

$$2x^2 - 4x - 16 = 0$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4 \text{ or } -2$$

$$A = \int_{-2}^4 (16 - x^2 - (x^2 - 4x)) dx$$

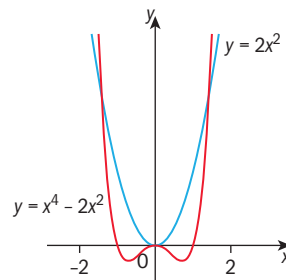
$$= \int_{-2}^4 (16 - 2x^2 + 4x) dx$$

$$= \left[16x - \frac{2}{3}x^3 + 2x^2 \right]_{-2}^4$$

$$= \left(64 - \frac{128}{3} + 32 \right) - \left(-32 + \frac{16}{3} + 8 \right)$$

$$= 72 \text{ sq. units}$$

6



$$x^4 - 2x^2 = 2x^2$$

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x^2(x-2)(x+2) = 0$$

$$x = 0, \pm 2$$

$$A = \int_{-2}^2 (2x^2 - (x^4 - 2x^2)) dx = \int_{-2}^2 (4x^2 - x^4) dx$$

$$= \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_{-2}^2 = \left(\frac{32}{3} - \frac{32}{5} \right) - \left(-\frac{32}{3} + \frac{32}{5} \right)$$

$$= \frac{128}{15} \text{ sq. units}$$

7 $2x^3 + 5x^2 + x - 2 = 8 - 4x - 20x^2 - 8x^3$

$$10x^3 + 25x^2 + 5x - 10 = 0$$

$$2x^3 + 5x^2 + x - 2 = 0$$

$$x = -2, -1, \frac{1}{2}$$

$$\text{Area} = \int_{-2}^{-1} (10x^3 + 25x^2 + 5x - 10) dx$$

$$+ \int_{-1}^{1/2} (-10x^3 - 25x^2 - 5x + 10) dx$$

$$= \left[\frac{5x^4}{2} + \frac{25x^3}{3} + \frac{5x^2}{2} - 10x \right]_{-2}^{-1}$$

$$+ \left[-\frac{5x^4}{2} - \frac{25x^3}{3} - \frac{5x^2}{2} + 10x \right]_{-1}^{1/2}$$

$$= \left(\frac{5}{2} - \frac{25}{3} + \frac{5}{2} + 10 \right) - \left(40 - \frac{200}{3} + 10 + 20 \right)$$

$$+ \left(-\frac{5}{32} - \frac{25}{24} - \frac{5}{8} + 5 \right) - \left(-\frac{5}{2} + \frac{25}{3} - \frac{5}{2} - 10 \right)$$

$$= \frac{20}{3} - \frac{10}{3} + \frac{305}{96} + \frac{20}{3} = \frac{1265}{96}$$

$$= 13.2 \text{ sq. units (3 sf)}$$

$$8 \quad x^4 - 4 = \frac{1}{1+x} \quad (x > 0)$$

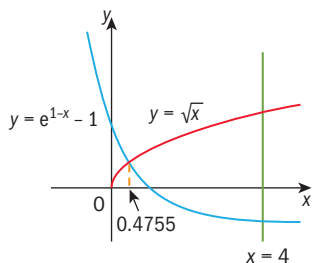
$$(x^4 - 4)(1 + x) = 1$$

$$x^5 + x^4 - 4x - 5 = 0$$

$$x = 1.449$$

$$A = \int_0^{1.449} \left(\frac{1}{1+x} - x^4 + 4 \right) dx = 5.41 \text{ sq. units}$$

9



$$A = \int_{0.4755}^4 (\sqrt{x} - e^{1-x} + 1) dx = 7.00 \text{ sq. units}$$

$$10 \quad y = x^2 \quad x = \sqrt{y}$$

$$\int_a^4 y^{\frac{1}{2}} dy = \int_0^a y^{\frac{1}{2}} dy$$

$$\left[\frac{2}{3} y^{\frac{3}{2}} \right]_a^4 = \left[\frac{2}{3} y^{\frac{3}{2}} \right]_0^a$$

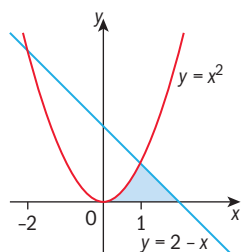
$$4^{\frac{3}{2}} - a^{\frac{3}{2}} = a^{\frac{3}{2}}$$

$$2a^{\frac{3}{2}} = 8$$

$$a^{\frac{3}{2}} = 4$$

$$a = 4^{\frac{2}{3}}$$

11



$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

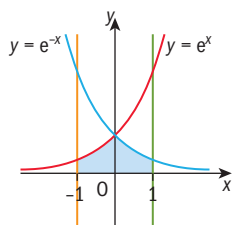
$$(x + 2)(x - 1) = 0$$

$$x = -2 \text{ or } 1$$

$$A = \int_{-2}^1 x^2 dx + \int_1^2 (2 - x) dx$$

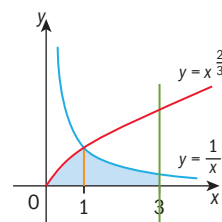
$$A = \left[\frac{x^3}{3} \right]_{-2}^1 + \left[2x - \frac{x^2}{2} \right]_1^2 = \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2} \right) = \frac{5}{6} \text{ sq. units}$$

12



$$\begin{aligned} \text{Area} &= 2 \int_0^1 e^{-x} dx \\ &= 2 [-e^{-x}]_0^1 \\ &= 2 (-e^{-1} + 1) = 2 \left(1 - \frac{1}{e} \right) \text{ or } 1.26 \text{ sq. units} \end{aligned}$$

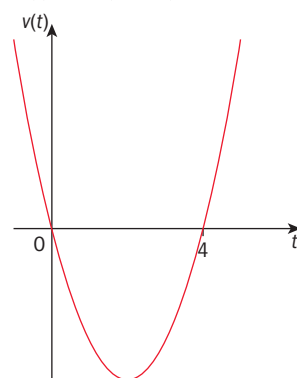
13



$$\begin{aligned} A &= \int_0^1 x^{\frac{2}{3}} dx + \int_1^3 \frac{1}{x} dx \\ &= \left[\frac{3}{5} x^{\frac{5}{3}} \right]_0^1 + [\ln |x|]_1^3 \\ &= \frac{3}{5} (1 - 0) + \ln 3 - \ln 1 \\ &= \frac{3}{5} + \ln 3 \text{ sq. units} \end{aligned}$$

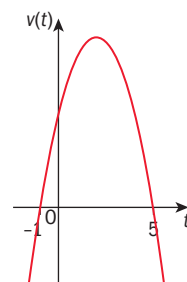
Exercise 7K

$$1 \quad v(t) = t(t - 4)$$



$$\begin{aligned} \text{distance} &= \left| \int_0^4 (t^2 - 4t) dt \right| \\ &= \left| \left[\frac{t^3}{3} - 2t^2 \right]_0^4 \right| \\ &= \left| \frac{64}{3} - 32 \right| = \left| \frac{-32}{3} \right| = \frac{32}{3} \text{ m} \end{aligned}$$

$$2 \quad v(t) = 5 + 4t - t^2 = (1 + t)(5 - t)$$



$$\begin{aligned} \text{a distance} &= \int_{-1}^5 (5 + 4t - t^2) dt \\ &= \left[5t + 2t^2 - \frac{t^3}{3} \right]_{-1}^5 \\ &= 5 + 2 - \frac{1}{3} = \frac{20}{3} \text{ m} \end{aligned}$$

$$\begin{aligned}
 \text{b distance} &= \int_1^5 (5 + 4t - t^2) dt \\
 &\quad + \left| \int_5^6 (5 + 4t - t^2) dt \right| \\
 &= \left[5t + 2t^2 - \frac{t^3}{3} \right]_1^5 + \left| \left[5t + 2t^2 - \frac{t^3}{3} \right]_5^6 \right| \\
 &= \left(25 + 50 - \frac{125}{3} \right) - \frac{20}{3} + \left| (30 + 72 - 72) - \left(25 + 50 - \frac{125}{3} \right) \right| \\
 &= \frac{80}{3} + \left| \frac{-10}{3} \right| = 30 \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 } a(t) &= 1 - e^{-2t} \quad 0 \leq t \leq 3 \\
 v(t) &= t + \frac{1}{2}e^{-2t} + c \\
 v(0) &= 0 \quad \therefore 0 = \frac{1}{2} + c \quad \therefore c = -\frac{1}{2} \\
 v(t) &= t + \frac{1}{2}e^{-2t} - \frac{1}{2} \\
 \text{distance} &= \int_0^3 \left(t + \frac{1}{2}e^{-2t} - \frac{1}{2} \right) dt \\
 &= \left[\frac{t^2}{2} - \frac{1}{4}e^{-2t} - \frac{1}{2}t \right]_0^3 = 3.25 \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 } v(t) &= 10 + 5e^{-0.5t} \\
 \text{a } a(t) &= -2.5e^{-0.5t} < 0 \quad \therefore \text{always negative} \\
 \text{b distance} &= \int_0^2 (10 + 5e^{-0.5t}) dt \\
 &= 26.3 \text{ m}
 \end{aligned}$$

Exercise 7L

$$\begin{aligned}
 \text{1 } y &= (x-1)^2 - 1 = x^2 - 2x \\
 v &= \pi \int_0^1 (x^2 - 2x)^2 dx = \pi \int_0^1 (x^4 - 4x^3 + 4x^2) dx \\
 &= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^1 \\
 &= \frac{8}{15} \pi \text{ cu. units or } 1.68 \text{ cu. units}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 } y &= 1 + \sqrt{x} \\
 v &= \pi \int_0^2 (1 + \sqrt{x})^2 dx = \pi \int_0^2 (1 + 2\sqrt{x} + x) dx \\
 &= \pi \left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2} \right]_0^2 \\
 &= \pi \left(2 + \frac{4}{3}(2\sqrt{2}) + 2 \right) = \pi \left(4 + \frac{4}{3}(2\sqrt{2}) \right) \\
 &= \frac{4\pi}{3} (3 + 2\sqrt{2}) \text{ cu. units or } 24.4 \text{ cu. units}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 } y &= \frac{x^2}{2} \quad x^2 = 2y \\
 v &= \pi \int_0^2 2y dy = \pi \left[y^2 \right]_0^2 = 4\pi \text{ cu. units}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 } y &= \sqrt{2x - x^2} \quad y^2 = 2x - x^2 \\
 v &= \pi \int_1^2 (2x - x^2) dx = \pi \left[x^2 - \frac{x^3}{3} \right]_1^2 \\
 &= \pi \left[\left(4 - \frac{8}{3} \right) - \left(1 - \frac{1}{3} \right) \right] = \frac{2\pi}{3} \text{ cu. units}
 \end{aligned}$$

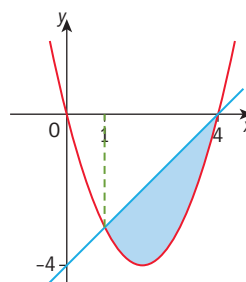
$$\begin{aligned}
 \text{5 } y &= x^{\frac{3}{2}} \quad x = y^{\frac{2}{3}} \quad x^2 = y^{\frac{4}{3}} \\
 v &= \pi \int_1^3 y^{\frac{4}{3}} dy = \pi \left[\frac{3}{7} y^{\frac{7}{3}} \right]_1^3 \\
 &= \frac{3\pi}{7} (3^{\frac{7}{3}} - 1) = 16.1 \text{ cu. units}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 } y &= \frac{x}{12} \sqrt{36 - x^2} \quad y^2 = \frac{x^2}{144} (36 - x^2) = \frac{x^2}{4} - \frac{x^4}{144} \\
 v &= \pi \int_0^6 \left(\frac{x^2}{4} - \frac{x^4}{144} \right) dx = \pi \left[\frac{x^3}{12} - \frac{x^5}{720} \right]_0^6 \\
 &= \pi (18 - 10.8) \\
 &= 7.2\pi \text{ cu. units}
 \end{aligned}$$

Exercise 7M

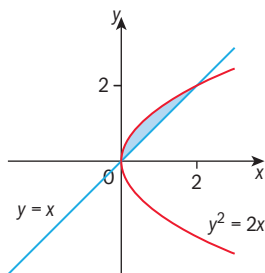
$$\begin{aligned}
 \text{1 } y &= x \quad y = \frac{x}{2} \\
 v &= \pi \int_2^5 \left(x^2 - \frac{x^4}{4} \right) dx = \pi \int_2^5 \frac{3x^2}{4} dx \\
 &= \pi \left[\frac{x^3}{4} \right]_2^5 = \pi \left(\frac{125}{4} - 2 \right) = \frac{117\pi}{4} \text{ cu. units}
 \end{aligned}$$

2



$$\begin{aligned}
 x - 4 &= x^2 - 4x \\
 x^2 - 5x + 4 &= 0 \\
 (x-1)(x-4) &= 0 \\
 x &= 1 \text{ or } 4 \\
 v &= \pi \int_1^4 ((x^2 - 4x)^2 - (x - 4)^2) dx \\
 &= \pi \int_1^4 (x^4 - 8x^3 + 16x^2 - x^2 + 8x - 16) dx \\
 &= \pi \int_1^4 (x^4 - 8x^3 + 15x^2 + 8x - 16) dx \\
 &= \pi \left[\frac{x^5}{5} - 2x^4 + 5x^3 + 4x^2 - 16x \right]_1^4 \\
 &= \pi \left[\left(\frac{1024}{5} - 512 + 320 + 64 - 64 \right) - \left(\frac{1}{5} - 2 + 5 + 4 - 16 \right) \right] \\
 &= \frac{108\pi}{5} \text{ cu. units}
 \end{aligned}$$

3



$$x^2 = 2x \quad x(x-2) = 0$$

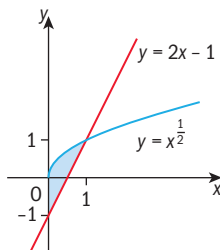
$$x = 0 \text{ or } 2 \quad y = 0 \text{ or } 2$$

$$v = \pi \int_0^2 \left(y^2 - \left(\frac{y^2}{2} \right)^2 \right) dy$$

$$v = \pi \int_0^2 \left(y^2 - \frac{y^4}{16} \right) dy$$

$$v = \pi \left[\frac{y^3}{3} - \frac{y^5}{20} \right]_0^2 = \pi \left(\frac{8}{3} - \frac{32}{20} \right) = \frac{16}{15} \pi \text{ cu. units}$$

4



$$y = 2x - 1 \quad y = x^2$$

$$x = \frac{y+1}{2} \quad x = y^2$$

$$x^2 = \frac{(y+1)^2}{4} \quad x^2 = y^4$$

$$v = \pi \int_{-1}^1 \frac{(y+1)^2}{4} dy - \pi \int_0^1 y^4 dy$$

$$= \pi \left[\frac{(y+1)^3}{12} \right]_{-1}^1 - \pi \left[\frac{y^5}{5} \right]_0^1$$

$$= \pi \left(\frac{8}{12} \right) - \pi \left(\frac{1}{5} \right) = \frac{7\pi}{15} \text{ cu. units}$$



Review exercise

1 $\frac{dy}{dx} = ax + \frac{b}{x^2}$ $(-1, 2)$ $(-2, 0)$ = stationary point

when $x = -2$, $\frac{dy}{dx} = 0 \quad \therefore -2a + \frac{b}{4} = 0$

$\therefore b = 8a$ (1)

$$y = \frac{ax^2}{2} - \frac{b}{x} + c$$

$(-2, 0) \quad 0 = 2a + \frac{b}{2} + c$ (2)

$(-1, 2) \quad 2 = \frac{a}{2} + b + c$ (3)

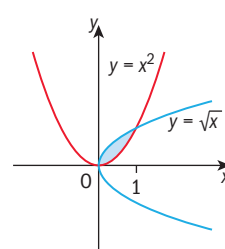
$(2) - (3) \quad -2 = \frac{3a}{2} - \frac{b}{2} \quad b = 8a$

$\therefore -2 = \frac{3a}{2} - 4a \Rightarrow -4 = 3a - 8a \Rightarrow -4 = -5a$

$$a = \frac{4}{5} \quad b = \frac{32}{5} \quad c = -2a - \frac{b}{2} = -\frac{8}{5} - \frac{16}{5} = -\frac{24}{5}$$

$$y = \frac{2}{5}x^2 - \frac{32}{5x} - \frac{24}{5}$$

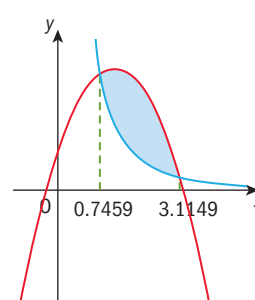
2



$$\text{Area} = \int_0^1 (x^{\frac{1}{2}} - x^2) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

3



$$v = \pi \int_{0.7459}^{3.1149} \left((1 + 3x - x^2)^2 - \left(\frac{2}{x} \right)^2 \right) dx$$

$$= 41.3 \text{ cu. units}$$

4 a $\int_1^2 \left(x + \frac{1}{x^2} - \frac{1}{x^4} \right) dx = \int_1^2 (x + x^{-2} - x^{-4}) dx$

$$= \left[\frac{x^2}{2} - x^{-1} + \frac{x^{-3}}{3} \right]_1^2 = \left[\frac{x^2}{2} - \frac{1}{x} + \frac{1}{3x^3} \right]_1^2$$

$$= \left(2 - \frac{1}{2} + \frac{1}{24} \right) - \left(\frac{1}{2} - 1 + \frac{1}{3} \right) = \frac{41}{24}$$

b $\int_1^4 \frac{5x^4 - 4}{\sqrt{x}} dx = \int_0^4 (5x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}) dx$

$$= \left[2x^{\frac{5}{2}} - 8x^{\frac{1}{2}} \right]_1^4 = (2(4)^{\frac{5}{2}} - 8(4)^{\frac{1}{2}}) - (2 - 8)$$

$$= (64 - 16) - (-6) = 54$$

c $\int_1^2 \frac{1}{x-3} dx = [\ln|x-3|]_1^2 = \ln 1 - \ln 2 = -\ln 2$

d $\int_1^e \frac{1}{1-4x} dx = -\frac{1}{4} [\ln|1-4x|]_1^e$

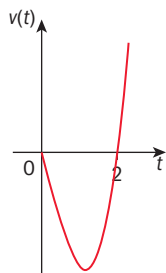
$$= -\frac{1}{4} (\ln|1-4e| - \ln 3)$$

$$= -\frac{1}{4} (\ln(4e-1) - \ln 3) = -\frac{1}{4} \ln \left(\frac{4e-1}{3} \right)$$



Review exercise

1 $v(t) = t^3 - 4t = t(t^2 - 4) = t(t - 2)(t + 2)$



$$\int_0^2 (t^3 - 4t) dt = \left[\frac{t^4}{4} - 2t^2 \right]_0^2$$

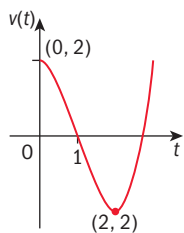
$$= 4 - 8 = -4$$

$$\int_2^3 (t^3 - 4t) dt = \left[\frac{t^4}{4} - 2t^2 \right]_2^3$$

$$= \left(\frac{81}{4} - 18 \right) - (-4) = \frac{25}{4}$$

$\therefore$ total distance $= 4 + \frac{25}{4} = \frac{41}{4}$ m

2 $v(t) = t^3 - 3t^2 + 2$



$$\int_0^1 (t^3 - 3t^2 + 2) dt = \left[\frac{t^4}{4} - t^3 + 2t \right]_0^1$$

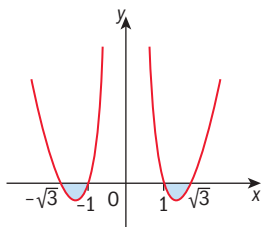
$$= \frac{1}{4} - 1 + 2 = \frac{5}{4}$$

$$\int_1^2 (t^3 - 3t^2 + 2) dt = \left[\frac{t^4}{4} - t^3 + 2t \right]_1^2$$

$$= (4 - 8 + 4) - \left(\frac{5}{4} \right) = \frac{-5}{4}$$

$\therefore$ total distance $= \frac{5}{4} + \frac{5}{4} = \frac{5}{2}$ m

3



$$y = x^2 - 4 + \frac{3}{x^2}$$

$$x^2 - 4 + \frac{3}{x^2} = 0$$

$$x^4 - 4x^2 + 3 = 0$$

$$(x^2 - 1)(x^2 - 3) = 0 \quad x = \pm 1, \pm\sqrt{3}$$

$$\int_1^{\sqrt{3}} \left(x^2 - 4 + \frac{3}{x^2} \right) dx = \left[\frac{x^3}{3} - 4x - \frac{3}{x} \right]_1^{\sqrt{3}}$$

$$= (\sqrt{3} - 4\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 4 - 3 \right) = -4\sqrt{3} + \frac{20}{3}$$

$\therefore$ total area $= 2 \left(4\sqrt{3} - \frac{20}{3} \right) = 8\sqrt{3} - \frac{40}{3}$ sq. units

4 a $\int \frac{3x^4 + 6}{x^2} dx = \int 3x^2 + \frac{6}{x^2} dx = x^3 - \frac{6}{x} + c$

b $\int \left(x + \frac{1}{x} \right) \left(x - \frac{1}{x} \right) dx = \int \left(x^2 - \frac{1}{x^2} \right) dx = \frac{x^3}{3} + \frac{1}{x} + c$

c $\int \frac{1}{2-3x} dx = -\frac{1}{3} \ln |2-3x| + c$

d $\int \frac{2}{\sqrt{1-4x}} dx = \int 2(1-4x)^{-\frac{1}{2}} dx = \frac{2}{-4} \frac{(1-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$$= -\sqrt{1-4x} + c$$

e $\int (2e^{-3x} + \sqrt[3]{e^x}) dx = \int (2e^{-3x} + e^{\frac{x}{3}}) dx$

$$= \frac{-2}{3} e^{-3x} + 3e^{\frac{x}{3}} + c = \frac{-2}{3} e^{-3x} + 3\sqrt[3]{e^x} + c$$

5 $2x - 1 \sqrt{2x^2 + 3x}$

$$\frac{2x^2 - x}{4x}$$

$$\frac{4x - 2}{2}$$

$\therefore \frac{2x^2 + 3x}{2x - 1} = x + 2 + \frac{2}{2x - 1}$

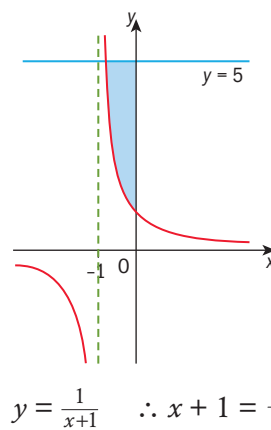
$$\int_1^2 \frac{2x^2 + 3x}{2x - 1} dx = \int_1^2 \left(x + 2 + \frac{2}{2x - 1} \right) dx$$

$$= \left[\frac{x^2}{2} + 2x + \ln |2x - 1| \right]_1^2$$

$$= (2 + 4 + \ln 3) - \left(\frac{1}{2} + 2 + \ln 1 \right)$$

$$= \frac{7}{2} + \ln 3$$

6



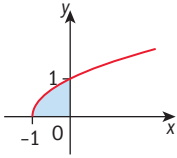
$y = \frac{1}{x+1} \quad \therefore x + 1 = \frac{1}{y}$

$$x = \frac{1}{y} - 1$$

$$\begin{aligned} \int_1^5 \left(\frac{1}{y} - 1 \right) dy &= [\ln y - y]_1^5 \\ &= (\ln 5 - 5) - (\ln 1 - 1) \\ &= \ln 5 - 4 \end{aligned}$$

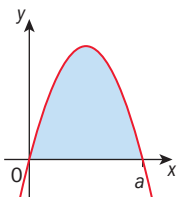
$$\therefore \text{area} = 4 - \ln 5 \text{ sq. units}$$

7



$$\begin{aligned} A &= \int_{-1}^0 (x+1)^{\frac{1}{2}} dx = \left[\frac{2}{3} (x+1)^{\frac{3}{2}} \right]_{-1}^0 \\ &= \frac{2}{3} \text{ sq. units} \end{aligned}$$

8



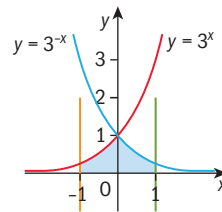
$$\int_0^a (3ax - 3x^2) dx = 4$$

$$\left[\frac{3ax^2}{2} - x^3 \right]_0^a = 4$$

$$\frac{3a^3}{2} - a^3 = 4$$

$$\frac{a^3}{2} = 4 \quad a^3 = 8 \quad \therefore a = 2$$

9



$$v = 2\pi \int_0^1 (3^{-x})^2 dx = 2\pi \int_0^1 3^{-2x} dx$$

$$= \frac{2\pi}{-2 \ln 3} [3^{-2x}]_0^1$$

$$= -\frac{\pi}{\ln 3} (3^{-2} - 1)$$

$$= \frac{\pi}{\ln 3} \left(\frac{8}{9} \right)$$

$$= \frac{8\pi}{9 \ln 3} \text{ cu. units}$$

8

Ancient mathematics
and modern methods

Answers

Skills check

- 1 Using Δ 's CFE and CBA, $\frac{EF}{6.5} = \frac{CF}{CB}$

Using Δ 's CDF and BAF, $\frac{4}{CF} = \frac{6.5}{BF}$

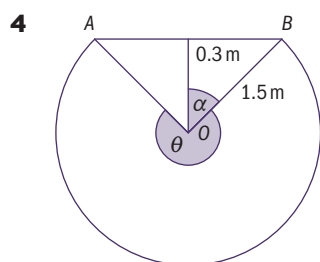
$$\frac{4}{CF} = \frac{6.5}{CB - CF}$$

$$\frac{CB - CF}{CF} = \frac{6.5}{4} \Rightarrow \frac{CB}{CF} - 1 = 1.625 \Rightarrow \frac{CB}{CF} = 2.265$$

$$\therefore \frac{EF}{6.5} = \frac{1}{2.265} \therefore EF = \frac{51}{21} = 2.48 \text{ m}$$

Exercise 8A

- 1 a $\hat{BAC} = 90^\circ - 28^\circ = 62^\circ$
 $\sin 28^\circ = \frac{AB}{8} \therefore AB = 8 \sin 28^\circ = 3.76 \text{ cm}$
 $\cos 28^\circ = \frac{BC}{8} \therefore BC = 8 \cos 28^\circ = 7.06 \text{ cm}$
 b $QR^2 = 7^2 - 4.2^2 \therefore QR = \sqrt{31.36} = 5.6 \text{ cm}$
 $\sin \hat{PRQ} = \frac{4.2}{7} \therefore \hat{PRQ} = 36.9^\circ$
 $\hat{QPR} = 90^\circ - 36.9^\circ = 53.1^\circ$
- 2 $\tan \alpha = \frac{BT}{AB} \therefore BT = 30 \tan 52.3^\circ = 38.8 \text{ m}$
- 3 $\cos \theta = \frac{r}{15} \therefore r = 15 \cos 1.23 = 5.013 \dots$
 arc length $= 2\pi r = 31.50 \dots$
 $\therefore 15\varphi = 31.50 \dots \therefore \varphi = 2.10 \text{ radians}$



$$\cos \alpha = \frac{0.3}{1.5} \therefore \alpha = 1.369 \dots$$

$$2\alpha = 2.738 \dots$$

$$\theta = 2\pi - 2.738 \dots = 3.544$$

$$\text{Area } \Delta AOB = \frac{1}{2}(1.5)(1.5)\sin 2.738 \dots$$

$$= 0.4409 \dots$$

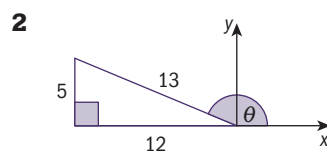
$$\text{Area major sector } AOB = \frac{1}{2}1.5^2(3.544 \dots) = 3.987 \dots$$

$$\begin{aligned} \text{Cross-sectional area of milk} &= 0.4409 \dots + 3.987 \dots \\ &= 4.428 \dots \end{aligned}$$

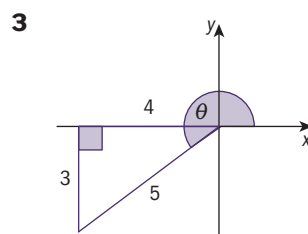
$$\therefore \text{Volume of milk} = 4.428 \dots \times 3 = 13.3 \text{ m}^3$$

Exercise 8B

- 1 a $\sin 144^\circ = \sin 36^\circ$
 b $\cos 210^\circ = -\cos 30^\circ$
 c $\tan 230^\circ = \tan 50^\circ$
 d $\sin\left(\frac{7\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$
 e $\tan\left(\frac{7\pi}{3}\right) = \tan \frac{\pi}{3}$
 f $\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$



$$\cos \theta = -\frac{12}{13} \quad \tan \theta = -\frac{5}{12}$$



$$\sec \theta = -\frac{5}{4} \Rightarrow \cos \theta = -\frac{4}{5}$$

$$\tan \theta = \frac{3}{4} \quad \sin \theta = -\frac{3}{5}$$

- 4 a $2 + 4\cos \theta$
 i max value = 6 when $\theta = 2\pi$
 ii min value = -2 when $\theta = \pi$
 b $5 - 3\sin \theta$
 i max value = 8 when $\theta = \frac{3\pi}{2}$
 ii min value = 2 when $\theta = \frac{\pi}{2}$
 c $2\sin \theta - 1$
 i max value = 1 when $\theta = \frac{\pi}{2}$
 ii min value = -3 when $\theta = \frac{3\pi}{2}$
 d $-2\cos \theta - 3$
 i max value = -1 when $\theta = \pi$
 ii min value = -5 when $\theta = 2\pi$

Investigation - trigonometric identities

- 1 a $\sin \theta = \cos(90^\circ - \theta)$, $\cos \theta = \sin(90^\circ - \theta)$,
 $\tan \theta = \frac{1}{\tan(90^\circ - \theta)}$
 b $\tan \theta = \frac{\sin \theta}{\cos \theta}$
 c $\sin^2 \theta + \cos^2 \theta = 1$
 e $\tan^2 \theta + 1 = \sec^2 \theta$
 f $\cot^2 \theta + 1 = \csc^2 \theta$

Investigation - exact values of sin, cos and tan

- 1 a $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\tan \frac{\pi}{4} = 1$
 b $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\cos \frac{\pi}{3} = \frac{1}{2}$ $\tan \frac{\pi}{3} = \sqrt{3}$
 $\sin \frac{\pi}{6} = \frac{1}{2}$ $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Exercise 8C

- 1 $\sin \theta = \frac{1}{4}$ $\frac{\pi}{2} \leq \theta \leq \pi$
 $\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{1}{16} = \frac{15}{16}$ $\therefore \cos \theta = \frac{-\sqrt{15}}{4}$
 $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-1}{\sqrt{15}}$
 2 $\cos \theta = \frac{-12}{13}$ $0 \leq \theta \leq \pi$ (θ lies in 2nd quadrant)
 $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{144}{169} = \frac{25}{169}$ $\therefore \sin \theta = \frac{5}{13}$
 $\tan \theta = \frac{\sin \theta}{\cos \theta} = -\frac{5}{12}$
 3 $\sin\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$
 $= \sin \frac{\pi}{6} = \frac{1}{2}$ (QED)
 4 $\frac{\sin \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin^2 \theta + (1 - \cos \theta)^2}{\sin \theta(1 - \cos \theta)}$
 $= \frac{\sin^2 \theta + 1 - 2 \cos \theta + \cos^2 \theta}{\sin \theta(1 - \cos \theta)}$
 $= \frac{2 - 2 \cos \theta}{\sin \theta(1 - \cos \theta)} = \frac{2(1 - \cos \theta)}{\sin \theta(1 - \cos \theta)} = \frac{2}{\sin \theta}$ (QED)
 5 $\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}$
 $= \frac{1}{\cos \theta \sin \theta} = \sec \theta \csc \theta$ (QED)
 $(\sin \theta + \cos \theta)(\tan \theta + \cot \theta) = (\sin \theta + \cos \theta) \sec \theta \csc \theta$
 $= \sin \theta \sec \theta \csc \theta + \cos \theta \sec \theta \csc \theta$
 $= \sec \theta + \csc \theta$ (QED)
 6 $\cot^2 \theta - \cos^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta} - \cos^2 \theta = \frac{\cos^2 \theta - \cos^2 \theta \sin^2 \theta}{\sin^2 \theta}$
 $= \frac{\cos^2 \theta(1 - \sin^2 \theta)}{\sin^2 \theta} = \frac{\cos^2 \theta \cos^2 \theta}{\sin^2 \theta}$
 $= \cos^4 \theta \csc^2 \theta$ (QED)

Exercise 8D

- 1 a $\sin 75^\circ = \sin(30^\circ + 45^\circ)$
 $= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$
 $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$
 $= \frac{1 + \sqrt{3}}{2\sqrt{2}}$
 b $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$
 $= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$
 c $\sec 105^\circ = \frac{1}{\cos(60^\circ + 45^\circ)} = \frac{1}{\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ}$
 $= \frac{1}{\frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{1 - \sqrt{3}} = -\frac{2\sqrt{2}}{\sqrt{3} - 1}$
 2 a $\cos 70^\circ \cos 10^\circ + \sin 70^\circ \sin 10^\circ = \cos 60^\circ = \frac{1}{2}$
 b $\frac{\tan 75^\circ}{\tan 15^\circ} = \frac{\tan(30^\circ + 45^\circ)}{\tan(45^\circ - 30^\circ)} = \frac{\frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}}{\frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}}$
 $= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} \times \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$
 $= \frac{(1 + \sqrt{3})^2}{(\sqrt{3} - 1)^2} = \left(\frac{(1 + \sqrt{3})^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)}\right)^2 = \frac{(1 + \sqrt{3})^4}{4}$
 3 $\sin \theta = \frac{24}{25}$ $0 < \theta < \frac{\pi}{2}$
 $\cos^2 \theta = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625}$ $\therefore \cos \theta = \frac{7}{25}$ $\tan \theta = \frac{24}{7}$
 $\sin \phi = \frac{3}{5}$ $\frac{\pi}{2} < \phi < \pi \Rightarrow \cos \phi = -\frac{4}{5}$ $\tan \phi = \frac{-3}{4}$
 $\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{\frac{24}{7} - \frac{3}{4}}{1 - \frac{24}{7} \times \frac{-3}{4}}$
 $= \frac{\frac{75}{28}}{\frac{75}{100}} = \frac{75}{100} = \frac{3}{4}$
 4 $\cot(A + B) = \frac{1}{\tan(A + B)} = \frac{1 - \tan A \tan B}{\tan A + \tan B} \times \frac{\frac{1}{\tan A \tan B}}{\frac{1}{\tan A \tan B}}$
 $= \frac{1}{\frac{1}{\tan A \tan B} + 1} - 1 = \frac{\cot A \cot B - 1}{\cot A + \cot B}$ (QED)
 5 a $\frac{\sin(A + B)}{\cos A \cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$
 $= \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \tan A + \tan B$ (QED)
 b $(\sin A + \cos A)(\sin B + \cos B)$
 $\equiv \sin A \sin B + \sin A \cos B + \cos A \sin B$
 $+ \cos A \cos B$
 $\equiv (\sin A \cos B + \cos A \sin B)$
 $+ (\sin A \sin B + \cos A \cos B)$
 $\equiv \sin(A + B) + \cos(A - B)$ (QED)

$$6 \quad a \quad \text{Let } \alpha = \arctan\left(\frac{1}{4}\right) \text{ and } \beta = \arctan\left(\frac{3}{5}\right)$$

$$\text{Then } \tan \alpha = \frac{1}{4} \text{ and } \tan \beta = \frac{3}{5}$$

$$\begin{aligned} \therefore \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \times \frac{3}{5}} = \frac{\frac{17}{20}}{\frac{17}{20}} = 1 \end{aligned}$$

$$\therefore \alpha + \beta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}.$$

$$\text{But } 0 < \alpha < \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \text{ so } 0 < \alpha + \beta < \frac{3\pi}{4}$$

$$\therefore \alpha + \beta = \frac{\pi}{4} \text{ i.e. } \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{3}{5}\right) = \frac{\pi}{4} \text{ (QED).}$$

$$b \quad \text{Let } \arctan(4) = \gamma$$

$$\text{Then } \tan \gamma = 4 = \frac{1}{\tan \alpha}$$

$$\therefore \gamma = \frac{\pi}{2} - 2$$

$$\text{Similarly if } \delta = \arctan\left(\frac{5}{3}\right), \text{ then } \delta = \frac{\pi}{2} - \beta$$

$$\begin{aligned} \therefore \arctan(4) + \arctan\left(\frac{5}{3}\right) &= \frac{\pi}{2} - \alpha + \frac{\pi}{2} - \beta \\ &= \pi - (\alpha + \beta) \\ &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

Exercise 8E

$$1 \quad a \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\text{Let } A = B = \theta$$

$$\sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta$$

$$\therefore \sin 2\theta = 2 \sin \theta \cos \theta \quad \text{(QED)}$$

$$b \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\text{Let } A = B = \theta$$

$$\cos 2\theta = \cos \theta \cos \theta - \sin \theta \sin \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{(QED)}$$

$$c \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\text{Let } A = B = \theta, \quad \tan 2\theta = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$$

$$\therefore \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \text{(QED)}$$

$$2 \quad \cos \alpha = \frac{4}{5} \quad \cos \beta = \frac{7}{25}$$

$$\sin \alpha = \pm \frac{3}{5} \quad \sin \beta = \pm \frac{24}{25}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{4}{5} \times \frac{7}{25} \pm \frac{3}{5} \times \frac{24}{25}$$

$$= \frac{100}{125} \text{ or } \frac{-44}{125} = \frac{4}{5} \text{ or } \frac{-44}{125}$$

$$3 \quad \cos A = \frac{1}{3}, \cos 2A = 2 \cos^2 A - 1 = 2\left(\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}$$

$$\cos 4A = 2 \cos^2 2A - 1 = 2\left(-\frac{7}{9}\right)^2 - 1 = \frac{17}{81}$$

$$\begin{aligned} 4 \quad \tan\left(\theta + \frac{\pi}{3}\right) \tan\left(\theta - \frac{\pi}{3}\right) &= \frac{(\tan \theta + \sqrt{3})(\tan \theta - \sqrt{3})}{(1 - \sqrt{3} \tan \theta)(1 + \sqrt{3} \tan \theta)} \\ &= \frac{\tan^2 \theta - 3}{1 - 3 \tan^2 \theta} \quad \text{(QED)} \end{aligned}$$

$$\begin{aligned} 5 \quad a \quad 2 \cos^2 A + \sin^2 A &= \cos^2 A + 1 = \frac{1}{2}(\cos 2A + 1) + 1 \\ &= \frac{1}{2}(\cos 2A + 3) \end{aligned}$$

$$b \quad \cos^4 A = (\cos^2 A)^2 = \frac{1}{4}(\cos 2A + 1)^2$$

$$c \quad \sin^4 A = (\sin^2 A)^2 = \frac{1}{4}(1 - \cos 2A)^2$$

$$\begin{aligned} 6 \quad a \quad (1 + \tan^2 \theta)(1 - \cos 2\theta) &= (1 + \tan^2 \theta)(1 - (1 - 2 \sin^2 \theta)) \\ &= (1 + \tan^2 \theta) 2 \sin^2 \theta \\ &= \sec^2 \theta 2 \sin^2 \theta \\ &= \frac{2 \sin^2 \theta}{\cos^2 \theta} \\ &= 2 \tan^2 \theta \quad \text{(QED)} \end{aligned}$$

$$\begin{aligned} b \quad (1 + \tan^2 \theta)(1 + \cos 2\theta) &= \sec^2 \theta (1 + 2 \cos^2 \theta - 1) \\ &= \frac{1}{\cos^2 \theta} (2 \cos^2 \theta) \\ &= 2 \quad \text{(QED)} \end{aligned}$$

$$7 \quad a \quad \frac{1 - \cos 2A}{1 + \cos 2A} = \frac{1 - (1 - 2 \sin^2 A)}{1 + (2 \cos^2 A - 1)} = \frac{2 \sin^2 A}{2 \cos^2 A} = \tan^2 A \quad \text{(QED)}$$

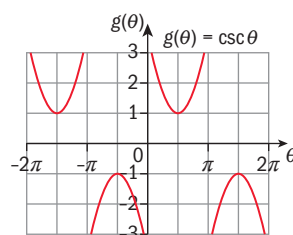
$$\begin{aligned} b \quad \frac{1 - \tan^2 A}{1 + \tan^2 A} &= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \\ &= \frac{\cos 2A}{1} = \cos 2A \quad \text{(QED)} \end{aligned}$$

$$\begin{aligned} c \quad \frac{\sin 2A}{1 - \cos 2A} &= \frac{2 \sin A \cos A}{1 - (1 - 2 \sin^2 A)} = \frac{2 \sin A \cos A}{2 \sin^2 A} \\ &= \frac{\cos A}{\sin A} = \cot A \quad \text{(QED)} \end{aligned}$$

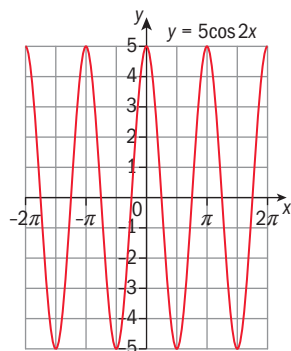
$$\begin{aligned} d \quad \cos 3A &= \cos(A + 2A) \\ &= \cos A \cos 2A - \sin A \sin 2A \\ &= \cos A (2 \cos^2 A - 1) - \sin A (2 \sin A \cos A) \\ &= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A \\ &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\ &= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A \\ &= 4 \cos^3 A - 3 \cos A \quad \text{(QED)} \end{aligned}$$

Exercise 8F

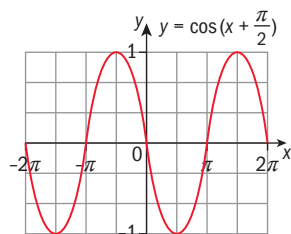
$$1 \quad a$$



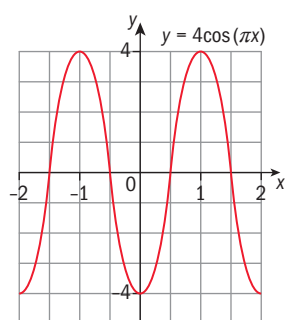
2 a



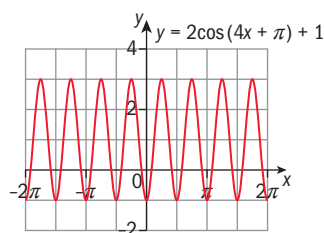
b



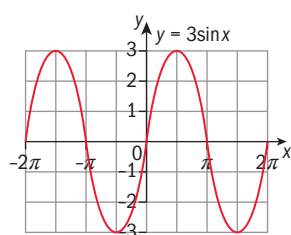
c



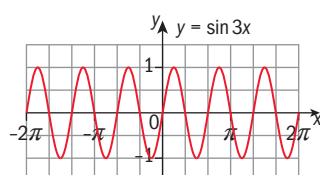
d



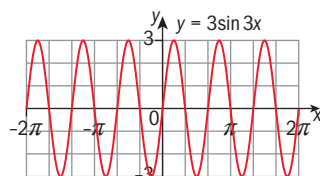
3 a



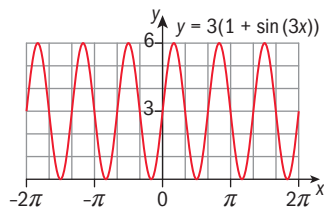
b



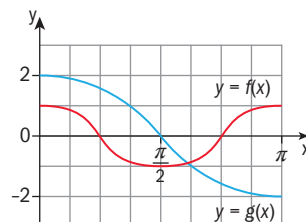
c



d



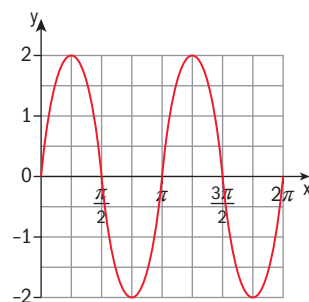
4



$f(x) = g(x)$, 1 solution

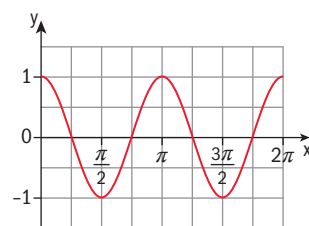
5 a $f(x) = 4 \sin x \cos x = 2 \sin 2x$

$f(x)$ is odd, period $= \pi$



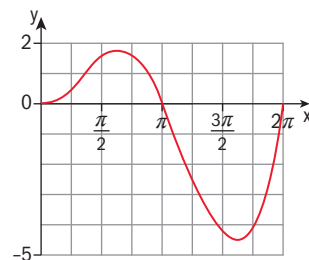
b $g(x) = 1 - 2 \sin^2 x = \cos^2 x$

$g(x)$ is even, period $= \pi$



c $h(x) = x \sin x$

$h(x)$ is even, not periodic



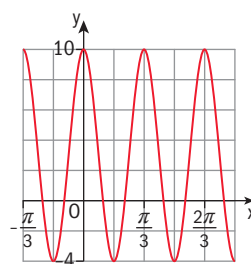
Exercise 8G

1 i $f(x) = 7 \sin \left[6 \left(x - \frac{\pi}{12} \right) \right] + 3$

a amplitude $= 7$ period $= \frac{2\pi}{6} = \frac{\pi}{3}$

phase shift $= \frac{\pi}{12}$

b min. value $= -4$, max. value $= 10$

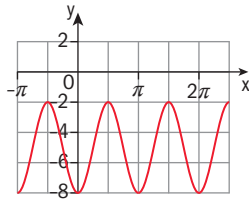


ii $f(x) = -3\sin\left(2x + \frac{\pi}{2}\right) - 5 = -3\sin\left[2\left(x + \frac{\pi}{4}\right)\right] - 5$

a amplitude = 3 period = $\frac{2\pi}{2} = \pi$

phase shift = $\frac{\pi}{4}$

b min value = -8 max value = -2

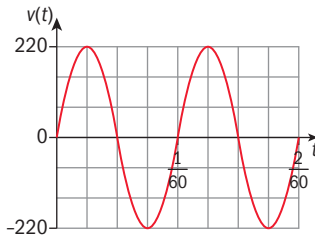


2 $V(t) = 220\sin(120\pi t)$

a max = 220 b min = -220

c amplitude = 220 d period = $\frac{2\pi}{120\pi} = \frac{1}{60}$

e $V(t)$



3 $h(t) = a\sin[b(t+c)] + d$

a $a = \frac{14.4 - 1.2}{2} = 6.6$

$d = \frac{14.4 + 1.2}{2} = 7.8$

$\frac{2\pi}{b} = 12 \therefore b = \frac{\pi}{6}$

$h(t) = 6.6\sin\left[\frac{\pi}{6}(t+c)\right] + 7.8$

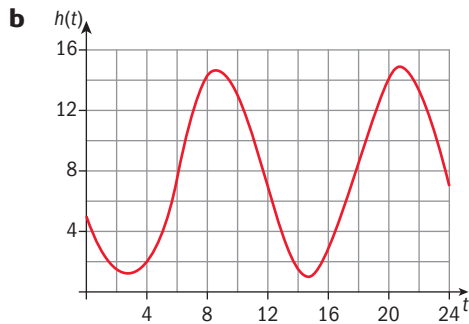
$h(8.25) = 14.4$

$\therefore 14.4 = 6.6\sin\left[\frac{\pi}{6}(8.25+c)\right] + 7.8$

$\therefore \sin\left[\frac{\pi}{6}(8.25+c)\right] = 1 \therefore \frac{\pi}{6}(8.25+c) = \frac{\pi}{2}$

$\therefore 8.25 + c = 3 \therefore c = -5.25$

$a = 6.6 \quad b = \frac{\pi}{6} \quad c = -5.25 \quad d = 7.8$



$h(t) = 6.6\sin\left[\frac{\pi}{6}(t - 5.25)\right] + 7.8$

$1.2 = 6.6\sin\left[\frac{\pi}{6}(t - 5.25)\right] + 7.8$

$\sin\left[\frac{\pi}{6}(t - 5.25)\right] = -1$

$\frac{\pi}{6}(t - 5.25) = \frac{-\pi}{2}$

$t - 5.25 = -3$

$\therefore t = 2.25$

First low tide is at 02:15

c Points of intersection: (0.086757, 5), (4.413243, 5), (12.086757, 5), (16.413243, 5)

$h(t) \geq 5$ for $0 \leq t \leq 0.086757$,

$4.413243 \leq t \leq 12.086757$, and $16.413243 \leq t \leq 24$

Time intervals are: 00:00 to 00:05, 04:25 to 12:05 and 16:25 to 24:00

4 $f(x) = a\sin[b(x+c)] + d$

$a = \frac{12.75 - 10.65}{2} = 1.05$

$d = \frac{12.75 + 10.65}{2} = 11.7$

$\frac{2\pi}{b} = 365 \therefore b = \frac{2\pi}{365}$

$f(x) = 1.05\sin\left[\frac{2\pi}{365}(x+c)\right] + 11.7$

On 21 June, $x = 172$, $f(172) = 12.75$

$12.75 = 1.05\sin\left[\frac{2\pi}{365}(172+c)\right] + 11.7$

$\sin\left[\frac{2\pi}{365}(172+c)\right] = 1 \therefore \frac{2\pi}{365}(172+c) = \frac{\pi}{2}$

$\therefore 172 + c = \frac{365}{4} \therefore c = -80.75$

$a = 1.05 \quad b = \frac{2\pi}{365} \quad c = -80.75 \quad d = 11.7$

On 4 July, $x = 185$, $f(185) = 12.72$

$\therefore 12.7$ hours

Exercise 8H

1 a $\cos\left(\arcsin\frac{\sqrt{2}}{2}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

b $\sec\left(\arctan\frac{1}{2}\right)$. Let $\theta = \arctan\frac{1}{2} \therefore \tan\theta = \frac{1}{2}$

$\sec^2\theta = 1 + \tan^2\theta = \frac{5}{4} \therefore \sec\theta = \frac{\sqrt{5}}{2}$

$\therefore \sec\left(\arctan\frac{1}{2}\right) = \frac{\sqrt{5}}{2}$

c $\cos\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$

d $\tan\left(\arctan\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

e $\arccos\left(\sin\left(\frac{3\pi}{4}\right)\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

f $\arcsin\left(\sin\left(\frac{-7\pi}{6}\right)\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$2 \quad \mathbf{a} \quad \sin\left(\arcsin\frac{1}{2} + \arccos\frac{1}{2}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \sin\frac{\pi}{2} = 1$$

$$\mathbf{b} \quad \text{Let } \arcsin\frac{3}{5} = \theta \quad \therefore \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$$

$$\arccos\left(\frac{-4}{5}\right) = \phi \quad \therefore \cos\phi = \frac{-4}{5}, \sin\phi = \frac{3}{5}$$

$$\cos\left(\arcsin\frac{3}{5} - \arccos\left(\frac{-4}{5}\right)\right) = \cos(\theta - \phi)$$

$$= \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$= \frac{4}{5} \times \frac{-4}{5} + \frac{3}{5} \times \frac{3}{5} = \frac{-7}{25}$$

$$\mathbf{c} \quad \text{Let } \arctan\frac{3}{4} = \theta, \quad \tan\theta = \frac{3}{4}$$

$$\tan\left(2\arctan\left(\frac{3}{4}\right)\right) = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

$$3 \quad \mathbf{a} \quad \text{Let } \arcsin a = \theta \quad \therefore \sin\theta = a \quad \cos\theta = \sqrt{1-a^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{\sqrt{1-a^2}}$$

$$\therefore \tan(\arcsin a) = \frac{a}{\sqrt{1-a^2}} \quad (\text{QED})$$

$$\mathbf{b} \quad \text{Let } \arcsin a = \theta \text{ and } \arccos a = \phi$$

$$\sin\theta = a \quad \cos\phi = a$$

$$\cos(\arcsin a + \arccos a) = \cos(\theta + \phi)$$

$$= \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$= \sqrt{1-a^2}(a) - a\sqrt{1-a^2}$$

$$= 0 \quad (\text{QED})$$

$$\mathbf{c} \quad \text{Let } \arccos a = \theta \quad \therefore \cos\theta = a, \sin\theta = \sqrt{1-a^2}$$

$$\tan(\arccos a) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-a^2}}{a} \quad (\text{QED})$$

Exercise 8I

$$1 \quad 3\sin x = 2\tan x \quad -\pi \leq x \leq \pi$$

$$3\sin x = \frac{2\sin x}{\cos x}$$

$$3\sin x \cos x - 2\sin x = 0$$

$$\sin x(3\cos x - 2) = 0$$

$$\sin x = 0 \text{ or } \cos x = \frac{2}{3}$$

$$x = 0, \pm\pi \text{ or } x = \pm 0.841$$

$$2 \quad \cot\theta + \sin\theta = 6 \quad [0, \pi]$$

$$\frac{\cos\theta}{\sin\theta} + \sin\theta = 6$$

$$\text{Using GDC: } \theta = 0.170$$

$$3 \quad 3\cos 2\theta = 2\cos^2\theta \quad [-\pi, \pi]$$

$$3(2\cos^2\theta - 1) = 2\cos^2\theta$$

$$4\cos^2\theta = 3 \quad \therefore \cos^2\theta = \frac{3}{4}, \cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$$

$$4 \quad 3\tan^2\theta - \frac{14}{\cos\theta} + 18 = 0$$

$$3(\sec^2\theta - 1) - 14\sec\theta + 18 = 0$$

$$3\sec^2\theta - 14\sec\theta + 15 = 0$$

$$(3\sec\theta - 5)(\sec\theta - 3) = 0$$

$$\sec\theta = \frac{5}{3} \text{ or } 3$$

$$5 \quad \sin x - \cos x = 1 \quad 0 \leq x \leq \pi$$

$$2\sin\frac{x}{2}\cos\frac{x}{2} - (2\cos^2\frac{x}{2} - 1) = 1$$

$$2\sin\frac{x}{2}\cos\frac{x}{2} - 2\cos^2\frac{x}{2} = 0$$

$$2\cos\frac{x}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) = 0$$

$$\cos\frac{x}{2} = 0 \text{ or } \sin\frac{x}{2} = \cos\frac{x}{2} \Rightarrow \tan\frac{x}{2} = 1$$

$$\frac{x}{2} = \frac{\pi}{2} \text{ or } \frac{x}{2} = \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{2} \text{ or } \pi$$

$$6 \quad \cos\theta + \sin\theta = 2 \quad -\pi \leq \theta \leq \pi$$

$$\frac{1}{\sin\theta} + \sin\theta = 2$$

$$1 + \sin^2\theta = 2\sin\theta$$

$$\sin^2\theta - 2\sin\theta + 1 = 0$$

$$(\sin\theta - 1)^2 = 0$$

$$\sin\theta = 1 \quad \therefore \theta = \frac{\pi}{2}$$

$$7 \quad \frac{\sin x - 3\cos x}{\sin x - \cos x} = 7$$

$$\sin x - 3\cos x = 7\sin x - 7\cos x$$

$$4\cos x = 6\sin x$$

$$\therefore \tan x = \frac{4}{6} = \frac{2}{3}$$

$$\mathbf{a} \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{\frac{4}{3}}{1-\frac{4}{9}} = \frac{4}{3} \times \frac{9}{5} = \frac{12}{5}$$

$$\mathbf{b} \quad \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$$\frac{2}{3} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$$2 - 2\tan^2\frac{x}{2} = 6\tan\frac{x}{2}$$

$$\tan^2\frac{x}{2} + 3\tan\frac{x}{2} - 1 = 0$$

$$\tan\frac{x}{2} = \frac{-3 \pm \sqrt{9 - (-4)}}{2} = \frac{-3 \pm \sqrt{13}}{2}$$

$$8 \quad \frac{x}{2}\sin 2x = \sqrt{x}\sin x \quad 0 \leq x \leq 2\pi$$

$$\text{From graph, } x = 0, \pi, 5.17, 2\pi$$

$$9 \quad -5x^2\cos 8x = \tan x \quad 0 \leq x \leq \frac{\pi}{2}$$

$$\text{Using GDC, } x = 0, 0.294, 0.536, 1.02, 1.32$$

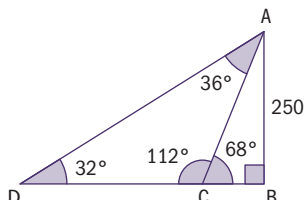
Exercise 8J

- 1 a $QR^2 = 5^2 + 8^2 - 2 \times 5 \times 8 \cos 30^\circ = 19.17...$
 $QR = 4.44$
 $\cos Q = \frac{5^2 + 4.44^2 - 8^2}{2 \times 5 \times 4.44} = -0.4342...$
 $\hat{PQR} = 11.6^\circ$ $\hat{PRQ} = 34.3^\circ$
- b $XZ^2 + 4^2 + 5^2 - 2 \times 4 \times 5 \cos 95^\circ = 44.48...$
 $XZ = 6.67$
 $\cos Z = \frac{4^2 + 6.67^2 - 5^2}{2 \times 4 \times 6.67} = 0.6650...$
 $\hat{XZY} = 48.3^\circ$ $\hat{YXZ} = 36.7^\circ$
- c $\cos A = \frac{4^2 + 8^2 - 5^2}{2 \times 4 \times 8} = 0.859375$
 $\hat{BAC} = 30.8^\circ$
 $\cos C = \frac{8^2 + 5^2 - 4^2}{2 \times 8 \times 5} = 0.9125$
 $\hat{ACB} = 24.1^\circ$ $\hat{ABC} = 125^\circ$
- 2 $\cos A = \frac{3.9^2 + 2.3^2 - 4.5^2}{2 \times 3.9 \times 2.3} = 0.01393...$
 $A = 89.2^\circ$, largest angle $= 89.2^\circ$
- 3 $\cos P = \frac{3^2 + 4^2 - 2^2}{2 \times 3 \times 4} = 0.875$
 $P = 29.0^\circ$, smallest angle $= 29.0^\circ$
- 4 $(2x - 1)^2 = x^2 + 5^2 - 2x^5 \cos 60^\circ$
 $4x^2 - 4x + 1 = x^2 + 25 - 5x$
 $3x^2 + x - 24 = 0$
 $(3x - 8)(x + 3) = 0$
 $x = \frac{8}{3}$
 $2x - 1 = \frac{13}{3}$, $\cos B = \frac{5^2 + (\frac{13}{3})^2 - (\frac{8}{3})^2}{2 \times 5 \times \frac{13}{3}} = 0.8461...$
 $\hat{ABC} = 32.2^\circ$ $\hat{ACB} = 87.8^\circ$
- 5 In $\triangle ABC$, $p^2 = a^2 + b^2 - 2ab \cos \hat{ABC}$
 In $\triangle ABD$, $q^2 = a^2 + b^2 - 2ab \cos \hat{BAD}$
 $\hat{BAD} = 180^\circ - \hat{ABC}$
 $\therefore \cos \hat{BAD} = \cos(180^\circ - \hat{ABC}) = -\cos \hat{ABC}$
 $\therefore q^2 = a^2 + b^2 + 2ab \cos \hat{ABC}$
 $\therefore p^2 + q^2 = 2(a^2 + b^2)$ (QED)

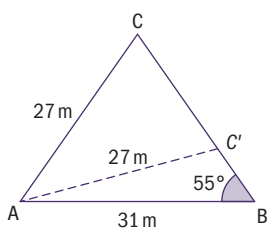
Exercise 8K

- 1 a $\hat{ACB} = 180^\circ - (30^\circ + 125^\circ) = 25^\circ$
 $\hat{ACB} = 25^\circ$
 $\frac{AC}{\sin 125^\circ} = \frac{10}{\sin 30^\circ} \therefore AC = 16.4 \text{ cm}$
 $\frac{AB}{\sin 25^\circ} = \frac{10}{\sin 30^\circ} \therefore AB = 8.45 \text{ cm}$
- b $\hat{PQR} = 95^\circ$
 $\frac{RP}{\sin 95^\circ} = \frac{7}{\sin 45^\circ} \therefore RP = 9.86 \text{ cm}$
 $\frac{QR}{\sin 95^\circ} = \frac{7}{\sin 45^\circ} \therefore QR = 6.36 \text{ cm}$

- c $\frac{\sin A}{7} = \frac{\sin 40^\circ}{9} \therefore \sin A = 0.4999...$
 $A = 29.996^\circ$ or 150.004° (only the acute angle is possible as this angle is opposite side 7 and therefore smaller than 40°)
 $\hat{BAC} = 30.0^\circ$ $\hat{ABC} = 110^\circ$
 $\frac{AC}{\sin 110^\circ} = \frac{9}{\sin 40^\circ} \therefore AC = 13.2 \text{ cm}$
- 2 $\frac{\sin Q}{80} = \frac{\sin 15^\circ}{150} \therefore \sin Q = 0.1380$
 $Q = 7.934^\circ$
 $R = 180^\circ - (15^\circ + 7.93^\circ) = 157.07^\circ$
 $\frac{PQ}{\sin 157.07^\circ} = \frac{150}{\sin 15^\circ} \therefore PQ = 225.84 \text{ km}$
 extra distance travelled $= 230 - 225.84$
 $= 4.16... \text{ km}$
 time lost $= \frac{4.16...}{400} \text{ hours} = 0.0104... \text{ hours}$
 $= 37 \text{ sec (nearest second)}$

- 3 
 In $\triangle ABC$,
 $\sin 68^\circ = \frac{250}{AC} \therefore AC = 269.63...$
 $\frac{CD}{\sin 36^\circ} = \frac{269.63...}{\sin 32^\circ}$
 $\therefore CD = 299 \therefore \text{length of lake} = 299 \text{ m (nearest m)}$

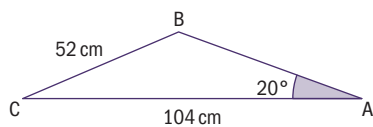
- 4 In $\triangle MBC$, $\hat{MBC} = 116^\circ$, $\hat{BMC} = 41^\circ$
 $\frac{MC}{\sin 116^\circ} = \frac{15}{\sin 41^\circ} \therefore MC = 20.5 \text{ m}$
 $\frac{MC}{\sin 23^\circ} = \frac{15}{\sin 41^\circ} \therefore MB = 8.93 \text{ m}$
 In $\triangle ABM$, $\sin 64^\circ = \frac{MA}{8.93...} \therefore MA = 8.03 \text{ m}$

- 5 
 $\frac{\sin C}{31} = \frac{\sin 55^\circ}{27} \therefore \sin C = 0.9405$
 $\therefore C = 70.1^\circ$ or 109.9°
 $A = 55^\circ$, $C = 70^\circ$ or $A = 15^\circ$, $C = 110^\circ$

Exercise 8L

- 1 $PR^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \cos 125^\circ = 418.129...$
 $PR = 20.448...$
 $\text{Area} = \frac{1}{2} \times 10 \times 13 \sin 125^\circ + \frac{1}{2} \times 15 \times 20.448 \sin 70^\circ$
 $= 197 \text{ sq. units}$

2



$$\frac{\sin B}{104} = \frac{\sin 20^\circ}{52}$$

$$\sin B = 0.68404\dots$$

$$B = 43.16\dots \text{ or } 136.83\dots$$

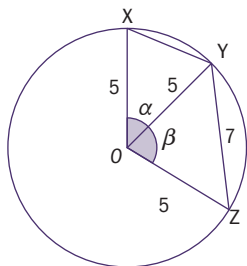
$$C = 116.83\dots \text{ or } 23.16\dots$$

$$\text{Area}_1 = \frac{1}{2} \times 52 \times 104 \sin 116.83\dots = 2412.7\dots$$

$$\text{Area}_2 = \frac{1}{2} \times 52 \times 104 \sin 23.16\dots = 1063.49\dots$$

$$\therefore \text{difference} = 1349 \text{ or } 1350 \text{ cm}^2 (3 \text{ sf})$$

3



$$\text{In } \triangle OXY, \cos \alpha = \frac{5^2 + 5^2 - 3^2}{2 \times 5 \times 5} = 0.82$$

$$\alpha = 34.915^\circ \dots$$

$$\text{In } \triangle OYZ, \cos \beta = \frac{5^2 + 5^2 - 7^2}{2 \times 5 \times 5} = 0.02$$

$$\beta = 88.854^\circ \dots$$

$$\begin{aligned} \text{Area OXYZ} &= \frac{1}{2} \times 5 \times 5 \sin 34.915^\circ + \frac{1}{2} \times 5 \times 5 \sin 88.854^\circ \\ &= 19.7 \text{ cm}^2 \end{aligned}$$

4 In $\triangle ABC$, $\tan 60^\circ = \frac{12}{AC} \therefore AC = 6.928$

In $\triangle ABD$, $\tan 55^\circ = \frac{12}{AD} \therefore AD = 8.402$

In $\triangle ACD$, $\cos \hat{CAD} = \frac{6.928^2 + 8.402^2 - 15^2}{2 \times 6.928 \times 8.402} = -0.9138\dots$

$$\therefore \hat{CAD} = 156^\circ$$

$$\text{Area } \triangle CAD = \frac{1}{2} \times 6.928 \times 8.402 \times \sin 156^\circ = 11.8 \text{ m}^2$$

5 a area $\triangle POQ = \frac{1}{2} r^2 \sin \frac{2\pi}{3} = \frac{1}{2} r^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} r^2$

$$\text{area } \triangle ROS = \frac{1}{2} r^2 \sin \frac{\pi}{6} = \frac{1}{4} r^2$$

b area $= \frac{1}{2} r^2 \frac{2\pi}{3} - \frac{\sqrt{13}}{4} r^2 = r^2 \left(\frac{\pi}{3} - \frac{\sqrt{13}}{4} \right) = \frac{r^2}{12} (4\pi - 3\sqrt{13})$

c area $= \frac{1}{2} r^2 \frac{\pi}{6} - \frac{1}{4} r^2 = r^2 \left(\frac{\pi}{12} - \frac{1}{4} \right) = \frac{r^2}{12} (\pi - 3)$

d shaded area $= \frac{r^2}{12} (4\pi - 3\sqrt{3}) - \frac{r^2}{12} (\pi - 3)$
 $= \frac{r^2}{12} (3\pi - 3\sqrt{3} + 3) = \frac{r^2}{4} (\pi + 1 - \sqrt{3}) \quad (\text{QED})$



Review exercise

1 $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2} \quad (\text{QED})$

$$\begin{aligned} \cos \theta &= 2 \cos^2 \frac{\theta}{2} - 1 = 2 \left(\frac{1}{\sqrt{1+t^2}} \right)^2 - 1 = \frac{2}{1+t^2} - 1 \\ &= \frac{2 - (1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2} \quad (\text{QED}) \end{aligned}$$

$$\sqrt{3} \sin \theta + \cos \theta = 1 \quad 0 \leq \theta \leq 2\pi$$

$$\frac{2\sqrt{3}t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$

$$2\sqrt{3}t + 1 - t^2 = 1 + t^2$$

$$2t^2 - 2\sqrt{3}t = 0$$

$$2t(t - \sqrt{3}) = 0$$

$$\tan \frac{\theta}{2} = 0 \text{ or } \tan \frac{\theta}{2} = \sqrt{3}$$

$$\frac{\theta}{2} = 0 \text{ or } \pi \text{ or } \frac{\theta}{2} = \frac{\pi}{3}$$

$$\theta = 0, \frac{2\pi}{3} \text{ or } 2\pi$$

2 a $\sin 165^\circ = \sin(120^\circ + 45^\circ)$
 $= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ$
 $= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

b $\tan 105^\circ = \tan(60^\circ + 45^\circ)$
 $= \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}$
 $= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$

c $\cos \frac{5\pi}{12} = \cos \left(\frac{\pi}{6} + \frac{\pi}{4} \right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$
 $= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

d $\tan \frac{\pi}{4} = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$
 $1 - \tan^2 \frac{\pi}{8} = 2 \tan \frac{\pi}{8}$
 $\tan^2 \frac{\pi}{8} + 2 \tan \frac{\pi}{8} - 1 = 0$
 $\tan \frac{\pi}{8} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 + \sqrt{2} \quad \text{since } \tan \frac{\pi}{8} > 0$
 $\therefore \tan \frac{\pi}{8} = \sqrt{2} - 1$

3 a $\frac{1}{1 - \tan \theta} = \frac{1}{1 - \frac{\sin \theta}{\cos \theta}} = \frac{\cos \theta}{\cos \theta - \sin \theta}$
 $\therefore \frac{\cos \theta}{\cos \theta - \sin \theta} = \frac{1}{1 - \tan \theta} \quad (\text{QED})$

b $\frac{\cos(A-B)}{\cos A \cos B} = \frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B}$
 $= \frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}$
 $= 1 + \tan A \tan B \quad (\text{QED})$

c $\cos 3A = \cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A$
 $= \cos A (2 \cos^2 A - 1) - \sin A 2 \sin A \cos A$
 $= 2 \cos^2 A - \cos A - 2 \sin^2 A \cos A$
 $\sin 3A = \sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$
 $= \sin A (2 \cos^2 A - 1) + \cos A 2 \sin A \cos A$
 $= 2 \sin A \cos^2 A - \sin A + 2 \sin A \cos^2 A$
 $= 4 \sin A \cos^2 A - \sin A$

$$\begin{aligned}\cos 3A - \sin 3A &= 2\cos^3 A - \cos A - 2\sin^2 A \cos A \\ &\quad - 4\sin A \cos^2 A + \sin A \\ &= 2\cos A(1 - \sin^2 A) - \cos A - 2\sin^2 A \cos A \\ &\quad - 4\sin A \cos^2 A + \sin A \\ &= \cos A - 4\sin^2 A \cos A - 4\sin A \cos^2 A + \sin A \\ &= \cos A(1 - 4\sin A \cos A) + \sin A(1 - 4\sin A \cos A) \\ &= (\cos A + \sin A)(1 - 4\sin A \cos A) \quad (\text{QED})\end{aligned}$$

$$\begin{aligned}\text{d } 2\sin 2\theta(1 - 2\sin^2 \theta) &= 2\sin 2\theta \cos 2\theta \\ &= \sin 4\theta \quad (\text{QED})\end{aligned}$$

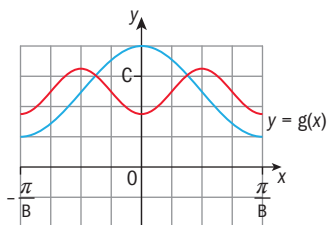
$$\begin{aligned}\text{e } 1 + 2\cos 2A + \cos 4A &= 1 + 2\cos 2A + 2\cos^2 2A - 1 \\ &= 2\cos 2A(1 + \cos 2A) \\ &= 2\cos 2A(1 + 2\cos^2 A - 1) \\ &= 4\cos^2 A \cos 2A \quad (\text{QED})\end{aligned}$$

$$\begin{aligned}\text{4 a Let } \arcsin \frac{3}{5} &= \theta, \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5} \\ \arccos \frac{1}{2} &= \phi, \cos \phi = \frac{1}{2}, \sin \phi = \frac{\sqrt{3}}{2} \\ \cos\left(\arcsin \frac{3}{5} - \arccos \frac{1}{2}\right) &= \cos(\theta - \phi) \\ &= \cos \theta \cos \phi + \sin \theta \sin \phi \\ &= \frac{4}{5} \times \frac{1}{2} + \frac{3}{5} \times \frac{\sqrt{3}}{2} = \frac{4 + 3\sqrt{3}}{10}\end{aligned}$$

$$\begin{aligned}\text{b Let } \arccos\left(\frac{-3}{5}\right) &= \theta \quad \therefore \cos \theta = \frac{-3}{5}, \sin \theta = \frac{4}{5} \\ \sin\left[2\arccos\left(\frac{-3}{5}\right)\right] &= \sin(2\theta) = 2\sin \theta \cos \theta \\ &= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \frac{-24}{25}\end{aligned}$$

$$\begin{aligned}\text{c } \arctan(-1) &= \frac{-\pi}{4} \\ \text{let } \arccos\left(\frac{-4}{5}\right) &= \theta \quad \therefore \cos \theta = \frac{-4}{5}, \sin \theta = \frac{3}{5} \\ \sin\left[\arctan(-1) + \arccos\left(\frac{-4}{5}\right)\right] &= \sin\left(\theta - \frac{\pi}{4}\right) \\ &= \sin \theta \cos \frac{\pi}{4} - \cos \theta \sin \frac{\pi}{4} \\ &= \frac{3}{5} \times \frac{\sqrt{2}}{2} + \frac{4}{5} \times \frac{\sqrt{2}}{2} \\ &= \frac{7\sqrt{2}}{10}\end{aligned}$$

5



$$\begin{aligned}\text{6 Let } \arcsin x &= \theta, \sin \theta = x, \cos \theta = \sqrt{1 - x^2} \\ \arccos x &= \phi, \cos \phi = x, \sin \phi = \sqrt{1 - x^2} \\ \sin[\arcsin x - \arccos x] &= \sin(\theta - \phi) \\ &= \sin \theta \cos \phi - \cos \theta \sin \phi \\ &= x^2 - (\sqrt{1 - x^2})^2 = x^2 - (1 - x^2) = 2x^2 - 1 \quad (\text{QED})\end{aligned}$$

$$\arcsin x - \arccos x = \arcsin(1 - x)$$

$$\therefore \sin(\arcsin(1 - x)) = 2x^2 - 1$$

$$1 - x = 2x^2 - 1$$

$$2x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 16}}{4}$$

$$\arcsin x \text{ acute} \Rightarrow x \geq 0$$

$$\arcsin(1 - x) \text{ acute} \Rightarrow 1 - x \geq 0$$

$$x = \frac{-1 + \sqrt{17}}{4}$$

$$\Rightarrow x \leq 1$$

$$\therefore 0 \leq x \leq 1$$

$$x = \frac{1}{4}(\sqrt{17} - 1) \quad (\text{QED})$$

$$\text{7 } \tan(2x + y) = \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y} = 1$$

$$\therefore \tan 2x + \tan y = 1 - \tan 2x \tan y$$

$$\tan y(1 + \tan 2x) = 1 - \tan 2x$$

$$\tan y = \frac{1 - \tan 2x}{1 + \tan 2x}$$

$$\tan y = \frac{1 - \frac{2 \tan x}{1 - \tan^2 x}}{1 + \frac{2 \tan x}{1 - \tan^2 x}} = \frac{1 - \tan^2 x - 2 \tan x}{1 - \tan^2 x + 2 \tan x}$$

$$\therefore \tan y = \frac{1 - 2 \tan x - \tan^2 x}{1 + 2 \tan x - \tan^2 x} \quad (\text{QED})$$



Review exercise

$$\begin{aligned}\text{1 } \cos(A - B) - \cos(A + B) &= \cos A \cos B + \sin A \sin B \\ &\quad - (\cos A \cos B - \sin A \sin B) \\ &= 2 \sin A \sin B \quad (\text{QED})\end{aligned}$$

$$\sin 3x \sin x = -1$$

$$\text{Let } A = 3x, B = x$$

$$\cos 2x - \cos 4x = 2 \sin 3x \sin x$$

$$\cos 2x - \cos 4x = -2$$

$$\cos 2x - (2 \cos^2 2x - 1) = -2$$

$$\cos 2x - 2 \cos^2 2x + 1 = -2$$

$$2 \cos^2 2x - \cos 2x - 3 = 0$$

$$(2 \cos 2x - 3)(\cos 2x + 1) = 0$$

$$\cos 2x = -1$$

$$2x = \pi$$

$$x = \frac{\pi}{2}$$

$$\text{2 a } \sin y + \sin x = 1.1 \Rightarrow y = \arcsin(1.1 - \sin x)$$

$$\cos y + \sin 2x = 1.8 \Rightarrow y = \arccos(1.8 - \sin 2x)$$

$$\text{b Using GDC, } x = 0.619, y = 0.546$$

$$\text{or } x = 1.09, y = 0.216$$

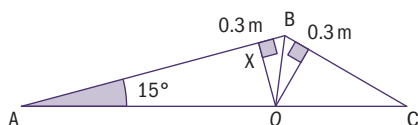
3 a $\hat{A}DB = 110^\circ \therefore \hat{A}BO = 180^\circ - (15^\circ + 110^\circ) = 55^\circ$

$$\hat{O}BC = \frac{1}{2}(180^\circ - 70^\circ) = 55^\circ$$

$$\therefore \hat{A}BC = 55^\circ + 55^\circ = 110^\circ$$

b In $\triangle ABC$, $\frac{AB}{\sin 55^\circ} = \frac{0.6}{\sin 15^\circ} \therefore AB = 1.90 \text{ m}$

c



$$AX = 1.898 - 0.3 = 1.598 \dots$$

$$\tan 15^\circ = \frac{OX}{1.598 \dots}$$

$$\therefore OX = 0.428 \text{ m}$$

$$\text{radius} = 0.428 \text{ m}$$

4 a $f(x) = \frac{2+3\sin x}{4+3\cos x}$, $0 \leq x \leq 2\pi$

For vertical asymptotes,

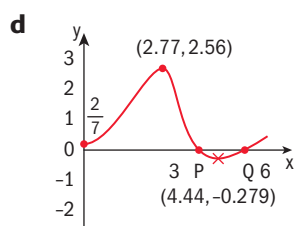
$$4 + 3\cos x = 0$$

$$\cos x = -\frac{4}{3} \text{ (no solution)}$$

$\therefore$ no vertical asymptotes (QED)

b $f(0) = \frac{2}{7}$, $\left(0, \frac{2}{7}\right)$

c $p = 3.87$ $q = 5.55$



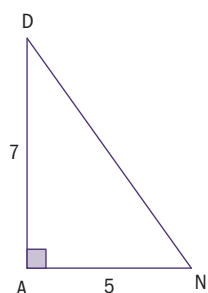
e Points of intersection at $x = 0.510$, 3.53 , 3.99 , 5.49

$$f(x) > g(x) \text{ for } 0.510 < x < 3.53$$

$$\text{and } 3.99 < x < 5.49$$

f Max. value of $f(x) - g(x)$ is 2.39
(when $x = 1.88$)

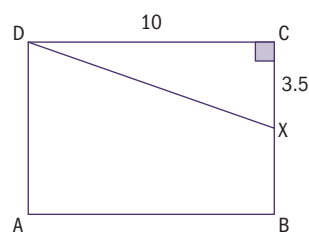
5 a



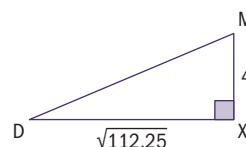
$$DN^2 = 5^2 + 7^2$$

$$DN = \sqrt{74} = 8.60 \text{ cm}$$

b



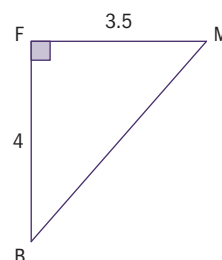
$$DX^2 = 10^2 + 3.5^2 = 112.25$$



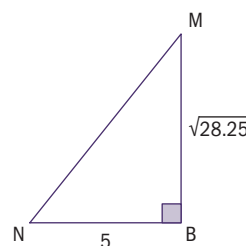
$$DM^2 = 112.25 + 4^2$$

$$DM = \sqrt{128.25} = 11.3 \text{ cm}$$

c



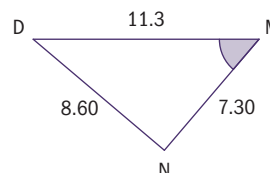
$$BM^2 = 3.5^2 + 4^2 = 28.25$$



$$NM^2 = 28.25 + 5^2$$

$$NM = \sqrt{53.25} = 7.30 \text{ cm}$$

d



$$\cos M = \frac{128.25 + 53.25 - 74}{2\sqrt{128.25}\sqrt{53.25}}$$

$$= 0.6504$$

$$\hat{D}MN = 49.4$$

e area $\triangle DMN = \frac{1}{2}\sqrt{128.25}\sqrt{53.25}\sin 49.4^\circ$
 $= 31.4 \text{ cm}^2$

6 a

$$\text{Area } \triangle ABC = \frac{1}{2}ch$$

(h = length of perpendicular from C to AB)

$$\text{or area } \triangle ABC = \frac{1}{2}ab\sin C$$

$$\therefore \frac{1}{2}ch = \frac{1}{2}ab\sin C$$

$$\therefore h = \frac{ab}{c}\sin C \text{ (QED)}$$

b In $\triangle BCD$, $\tan 30^\circ = \frac{10}{BC} \quad \therefore BC = 17.3 \text{ m}$

In $\triangle ACD$, $\tan 45^\circ = \frac{10}{AC} \quad \therefore AC = 10 \text{ m}$

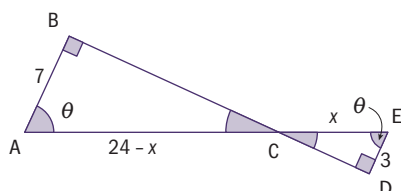
In $\triangle ABC$, $AB^2 = 17.3^2 + 10^2 - 2 \times 17.3$
 $\times 10 \cos 150^\circ = 700$

$AB = 26.5 \text{ m}$

From a , $h = \frac{ab}{c} \sin C = \frac{17.3 \times 10}{26.5} \sin 150^\circ$

$h = 3.27 \text{ m}$

7



$$\frac{3}{7} = \frac{x}{24-x}$$

$$72 - 3x = 7x$$

$$x = 7.2 \text{ cm}$$

In $\triangle CDE$, $CD^2 = 7.2^2 - 3^2 = 42.84$,

$$CD = \sqrt{42.84}$$

In $\triangle ABC$, $BC^2 = 16.8^2 - 7^2 = 233.24$,

$$BC = \sqrt{233.24}$$

$$\cos \theta = \frac{7}{16.8} \quad \therefore \theta = 1.141$$

Major arc of large circle $= 7(2\pi - 2\theta) = 28.008\dots$

Major arc of small circle $= 3(2\pi - 2\theta) = 12.003\dots$

Length of belt $= 2BC + 2CD + 28.008 + 12.003$
 $= 83.6 \text{ cm}$

9

The power of calculus

Answers

Skills check

$$1 \quad a \quad \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{1} = \cos^2 \theta - \sin^2 \theta$$

$$\therefore \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$b \quad \tan 2\theta = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\therefore \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$2 \quad a \quad f(x) = 3e^{2x} - 2x^2, \quad f'(x) = 6e^{2x} - 4x$$

$$b \quad g(x) = (x+1) \ln(x^2 + 2x + 1)$$

$$= (x+1) \ln(x+1)^2 = 2(x+1) \ln(x+1)$$

$$g'(x) = \frac{2(x+1)}{(x+1)} + 2 \ln(x+1) = 2 + 2 \ln(x+1)$$

$$c \quad h(x) = \frac{e^{x^2}}{x+1}, \quad h'(x) = \frac{(x+1)2xe^{x^2} - e^{x^2}}{(x+1)^2}$$

$$= \frac{e^{x^2}(2x^2 + 2x - 1)}{(x+1)^2}$$

Exercise 9A

Proofs using differentiation from first principles

Exercise 9B

$$1 \quad a \quad y = \cot x \quad \frac{dy}{dx} = -\csc^2 x$$

$$b \quad y = \csc x \quad \frac{dy}{dx} = -\csc x \cot x$$

$$c \quad y = \sin 3x \quad \frac{dy}{dx} = 3 \cos 3x$$

$$d \quad y = \tan(5x - 3) \quad \frac{dy}{dx} = 5 \sec^2(5x - 3)$$

$$e \quad y = \cos(8 - 3x) \quad \frac{dy}{dx} = 3 \sin(8 - 3x)$$

$$f \quad y = \csc\left(\frac{x-3}{4}\right) \quad \frac{dy}{dx} = -\frac{1}{4} \csc\left(\frac{x-3}{4}\right) \cot\left(\frac{x-3}{4}\right)$$

$$g \quad y = \cot\left(\frac{7-2x}{13}\right) \quad \frac{dy}{dx} = \frac{2}{13} \csc^2\left(\frac{7-2x}{13}\right)$$

$$2 \quad a \quad y = \sin(x^5 - 3) \quad \frac{dy}{dx} = 5x^4 \cos(x^5 - 3)$$

$$b \quad y = \cos(e^x) \quad \frac{dy}{dx} = -e^x \sin(e^x)$$

$$c \quad y = \csc(x^2 + 11) \quad \frac{dy}{dx} = -2x \csc(x^2 + 11) \cot(x^2 + 11)$$

$$d \quad y = \cot(4x^3 - 2x^2 + 7x + 17)$$

$$\frac{dy}{dx} = -(12x^2 - 4x + 7) \csc^2(4x^3 - 2x^2 + 7x + 17)$$

$$e \quad y = \tan(\ln(2x + 1)), \quad \frac{dy}{dx} = \frac{2}{2x + 1} \sec^2(\ln(2x + 1))$$

$$f \quad y = \sec(\sqrt{e^x + 1})$$

$$\frac{dy}{dx} = \frac{1}{2} e^x (e^x + 1)^{-\frac{1}{2}} \sec(\sqrt{e^x + 1}) \tan(\sqrt{e^x + 1})$$

$$= \frac{e^x \sec(\sqrt{e^x + 1}) \tan(\sqrt{e^x + 1})}{2(\sqrt{e^x + 1})}$$

$$= \frac{e^x \sin(\sqrt{e^x + 1}) \sec^2(\sqrt{e^x + 1})}{2(\sqrt{e^x + 1})}$$

$$g \quad y = \sin(\cos(\tan x))$$

$$\frac{dy}{dx} = -\sec^2 x \sin(\tan x) \cos(\cos(\tan x))$$

Exercise 9C

$$1 \quad a \quad y = (2x - 1) \cos x$$

$$\frac{dy}{dx} = 2 \cos x - (2x - 1) \sin x$$

$$b \quad y = (3x - x^2) \sin 2x$$

$$\frac{dy}{dx} = (3 - 2x) \sin 2x + 2(3x - x^2) \cos 2x$$

$$c \quad y = e^{1-x} \tan x$$

$$\frac{dy}{dx} = e^{1-x} \sec^2 x - e^{1-x} \tan x = e^{1-x} (\sec^2 x - \tan x)$$

$$d \quad y = \frac{\sin x}{x}, \quad \frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$$

$$e \quad y = \frac{2x + 3}{\sin 2x}, \quad \frac{dy}{dx} = \frac{2 \sin 2x - 2(2x + 3) \cos 2x}{\sin^2 2x}$$

$$f \quad y = \frac{\tan x}{\sqrt{2-x}}$$

$$\frac{dy}{dx} = \left(\sqrt{2-x} \sec^2 x + \frac{1}{2} (2-x)^{-\frac{1}{2}} \tan x \times \frac{1}{(2-x)} \right)$$

$$\frac{dy}{dx} = \frac{2(2-x) \sec^2 x + \tan x}{2(2-x)^{\frac{3}{2}}}$$

2 a $y = \sin 2x \quad x = \frac{\pi}{6}$

$$\frac{dy}{dx} = 2\cos 2x = 2\cos \frac{\pi}{3} = 1$$

b $y = \cos 3x \quad x = \frac{7\pi}{12}$

$$\frac{dy}{dx} = -3\sin 3x = -3\sin \frac{7\pi}{4} = \frac{3}{\sqrt{2}}$$

c $y = \tan(-x) = -\tan x \quad x = \frac{5\pi}{4}$

$$\frac{dy}{dx} = -\sec^2 x = -\sec^2 \frac{5\pi}{4} = -2$$

d $y = (x-2)\sin x \quad x = 0$

$$\begin{aligned} \frac{dy}{dx} &= \sin x + (x-2)\cos x \\ &= \sin 0 + (-2)\cos 0 = -2 \end{aligned}$$

e $y = -3x\cos x \quad x = \frac{\pi}{2}$

$$\begin{aligned} \frac{dy}{dx} &= -3\cos x + 3x\sin x \\ &= -3\cos \frac{\pi}{2} + 3\frac{\pi}{2}\sin \frac{\pi}{2} = \frac{3\pi}{2} \end{aligned}$$

f $y = x^2 \tan x \quad x = \frac{3\pi}{4}$

$$\begin{aligned} \frac{dy}{dx} &= 2x \tan x + x^2 \sec^2 x \\ &= \frac{3\pi}{2} \tan\left(\frac{3\pi}{4}\right) + \frac{9\pi^2}{16} \sec^2 \frac{3\pi}{4} \\ &= -\frac{3\pi}{2} + \frac{9\pi^2}{8} \end{aligned}$$

g $y = e^x \sec x \quad x = 0 \quad \frac{dy}{dx} = e^x \sec x \tan x + e^x \sec x$
 $= \sec 0 \tan 0 + \sec 0 = 1$

3 a $y = \sin^2 \alpha + \cos^2 \alpha = 1 \quad \frac{dy}{d\alpha} = 0$

b $y = \frac{\tan \beta}{\sin \beta} = \sec \beta \quad \frac{dy}{d\beta} = \sec \beta \tan \beta$

c $y = \frac{2 \tan 2\theta}{1 - \tan^2 \theta} = \tan 4\theta \quad \frac{dy}{d\theta} = 4 \sec^2 4\theta$

d $y = \frac{\sin \rho + \sin 2\rho}{\cos \rho + \cos 2\rho} = \frac{2 \sin \frac{3\rho}{2} \cos \frac{\rho}{2}}{2 \cos \frac{3\rho}{2} \cos \frac{\rho}{2}}$

$$y = \tan \frac{3\rho}{2} \quad \frac{dy}{d\rho} = \frac{3}{2} \sec^2 \frac{3\rho}{2}$$

e $y = \frac{(\sin \phi \sin 2\phi - \cos \phi) \sec \phi}{\sin \phi - \cos \phi}$
 $= \frac{2 \sin^2 \phi - 1}{\sin \phi - \cos \phi} = \frac{\sin^2 \phi - \cos^2 \phi}{\sin \phi - \cos \phi}$
 $= \frac{(\sin \phi - \cos \phi)(\sin \phi + \cos \phi)}{\sin \phi - \cos \phi}$

$$y = \sin \phi + \cos \phi \quad \frac{dy}{d\phi} = \cos \phi - \sin \phi$$

Exercise 9D

1 a $y = \arccos x \therefore x = \cos y$

$$\begin{aligned} \frac{dx}{dy} &= -\sin y \therefore \frac{dx}{dy} = -\frac{1}{\sin y} \\ &= -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}} \\ \therefore f'(x) &= -\frac{1}{\sqrt{1-x^2}} \end{aligned}$$

b $f(x) = \arcsin 3x, f'(x) = \frac{3}{\sqrt{1-9x^2}}$

c $f(x) = \arctan(2x+1), f'(x) = \frac{2}{1+(2x+1)^2}$
 $= \frac{2}{1+4x^2+4x+1}$
 $f'(x) = \frac{1}{2x^2+2x+1}$

2 a $y = 2x \arcsin x$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^2}} + 2 \arcsin x$$

b $y = \frac{\arccos x}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x}{\sqrt{1-x^2} - \arccos x} \\ \frac{dy}{dx} &= \frac{-x - \sqrt{1-x^2} \arccos x}{x^2 \sqrt{1-x^2}} \end{aligned}$$

c $y = (2x+1) \arctan x$

$$\frac{dy}{dx} = 2 \arctan x + \frac{2x+1}{1+x^2}$$

d $y = \sqrt{1-x^2} \arcsin x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - 2x \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \arcsin x \\ \frac{dy}{dx} &= 1 - \frac{x \arcsin x}{\sqrt{1-x^2}} \end{aligned}$$

e $y = (4x^2+1) \arctan 2x$

$$\begin{aligned} \frac{dy}{dx} &= 8x \arctan 2x + \frac{2(4x^2+1)}{1+4x^2} \\ \frac{dy}{dx} &= 8x \arctan 2x + 2 \end{aligned}$$

3 a $\frac{d}{dx} (\arcsin x + \arccos x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

b $\frac{d}{dx} (\arctan x + \arctan(-x)) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

$$\begin{aligned} \text{c } \frac{d}{dx} \left(2 \arctan x - \arcsin \frac{2x}{x^2+1} \right) \\ = \frac{2}{1+x^2} - \left(\frac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2} \right) \frac{1}{\sqrt{1-\frac{4x^2}{(x^2+1)^2}}} \\ = \frac{2}{1+x^2} - \frac{(2-2x^2)}{(x^2+1)^2} \frac{(x^2+1)}{\sqrt{(x^2+1)^2-4x^2}} \\ = \frac{2}{1+x^2} - \frac{2(1-x^2)}{(x^2+1)\sqrt{(x^2-1)^2}} = 0 \end{aligned}$$

4 a $x = \sin y$

$$1 = \cos y \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

b $x + y = \tan y$

$$1 + \frac{dy}{dx} = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} (\sec^2 y - 1) = 1 \Rightarrow \frac{dy}{dx} \tan^2 y = 1 \Rightarrow \frac{dy}{dx} = \cot^2 y$$

c $x + \sin x = y + \cos y$

$$1 + \cos x = \frac{dy}{dx} (1 - \sin y) \quad \therefore \frac{dy}{dx} = \frac{1 + \cos x}{1 - \sin y}$$

d $e^{\sin y} = x^2$

$$e^{\sin y} \cos y \frac{dy}{dx} = 2x \quad \therefore \frac{dy}{dx} = \frac{2x}{e^{\sin y} \cos y}$$

e $\cos y = \frac{x}{y} \Rightarrow y \cos y = x$

$$\frac{dy}{dx} (\cos y - y \sin y) = 1 \quad \therefore \frac{dy}{dx} = \frac{1}{\cos y - y \sin y}$$

f $\ln(xy) = \tan 2y \Rightarrow \ln x + \ln y = \tan 2y$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 2 \sec^2 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(2 \sec^2 2y - \frac{1}{y} \right) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}}{2 \sec^2 2y - \frac{1}{y}} = \frac{y}{2xy \sec^2 2y - x}$$

Exercise 9E

1 a $f(x) = \tan 3x$ P(0, 0)

$$f'(x) = 3 \sec^2 3x \quad f'(0) = 3$$

$$y = 3x$$

b $f(x) = \sin(2x) - 1$ P $\left(\frac{\pi}{3}, y\right)$

$$y = \sin\left(\frac{2\pi}{3}\right) - 1 = \frac{\sqrt{3}}{2} - 1 \quad \text{P}\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2} - 1\right)$$

$$f'(x) = 2 \cos(2x) \quad f'\left(\frac{\pi}{3}\right) = -1$$

$$y - \frac{\sqrt{3}}{2} + 1 = -1 \left(x - \frac{\pi}{3} \right)$$

$$y = -x + \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$$

c $f(x) = 2 \cos\left(\frac{x}{2}\right) - e^{2x}$ P(0, 1)

$$f'(x) = -\sin\left(\frac{x}{2}\right) - 2e^{2x} \quad \therefore f'(0) = -2$$

$$y = -2x + 1$$

d $f(x) = \ln\left(\tan\left(\frac{x}{3}\right)\right) + 2$ P $\left(\frac{3\pi}{4}, y\right)$

$$y = \ln\left(\tan\left(\frac{\pi}{4}\right)\right) + 2 = 2 \quad \text{P}\left(\frac{3\pi}{4}, 2\right)$$

$$f'(x) = \frac{\frac{1}{3} \sec^2\left(\frac{x}{3}\right)}{\tan\left(\frac{x}{3}\right)} \quad f'\left(\frac{3\pi}{4}\right) = \frac{1}{3} \sec^2\left(\frac{\pi}{4}\right) = \frac{2}{3}$$

$$y - 2 = \frac{2}{3} \left(x - \frac{3\pi}{4} \right) \quad y = \frac{2}{3}x - \frac{\pi}{2} + 2$$

2 a $f(x) = \cos(2x)$ P(0, 1)

$$f'(x) = -2 \sin(2x), \quad f'(0) = 0$$

equation of normal is $x = 0$

b $f(x) = \tan(4x)$ P $\left(\frac{\pi}{16}, y\right)$

$$y = \tan\left(\frac{\pi}{4}\right) = 1 \quad \text{P}\left(\frac{\pi}{16}, 1\right)$$

$$f'(x) = 4 \sec^2(4x), \quad f'\left(\frac{\pi}{16}\right) = 4 \sec^2\left(\frac{\pi}{4}\right) = 8$$

$$y - 1 = \frac{-1}{8} \left(x - \frac{\pi}{16} \right)$$

$$y = \frac{-1}{8}x + \frac{\pi}{128} + 1$$

c $f(x) = 2e^x \sin\left(\frac{x}{2}\right)$ P(0, y) $y = 0$ P(0, 0)

$$f'(x) = e^x \cos\left(\frac{x}{2}\right) + 2e^x \sin\left(\frac{x}{2}\right)$$

$$f'(0) = 1 \quad y = -x$$

d $f(x) = x \cos(2x) - 3$ P $\left(\frac{\pi}{2}, y\right)$

$$y = \frac{\pi}{2} \cos \pi - 3 = -\frac{\pi}{2} - 3 \quad \text{P}\left(\frac{\pi}{2}, -\frac{\pi}{2} - 3\right)$$

$$f'(x) = -2x \sin(2x) + \cos(2x)$$

$$f'\left(\frac{\pi}{2}\right) = -1 \quad y + \frac{\pi}{2} + 3 = x - \frac{\pi}{2}$$

$$y = x - \pi - 3$$

3 $\ln(x) = \tan y$ P(1, 0)

$$\frac{1}{x} = \sec^2 y \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{\cos^2 y}{x}$$

$$\text{At P, } \frac{dy}{dx} = 1 \quad y = x - 1$$

4 $y + y^2 = \sin 2x$ P(0, -1)

$$\frac{dy}{dx}(1 + 2y) = 2\cos 2x$$

$$\frac{dy}{dx} = \frac{2\cos 2x}{1 + 2y} = \frac{2}{-1} = -2$$

$$y + 1 = \frac{1}{2}x \quad y = \frac{1}{2}x - 1$$

5 $y = \cos(x^2)$

a (0.974, 0.583)

b $y = \cos(x^2)$

$$\frac{dy}{dx} = -2x \sin(x^2)$$

$$x = 0.97407123, \frac{dy}{dx} = -1.5833 \dots$$

$$y - 0.58264678 = -1.5833(x - 0.97407123)$$

$$y = -1.58x + 2.12$$

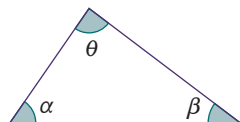
$$y = e^{x^2} - 2$$

$$\frac{dy}{dx} = 2xe^{x^2} = 5.03136 \dots$$

$$y - 0.58264678 = 5.03136(x - 0.97407123)$$

$$y = 5.03x - 4.32$$

c



$$\tan \alpha = 5.03136 \dots$$

$$\alpha = 1.3746$$

$$\tan \beta = 1.5833 \dots$$

$$\beta = 1.00747$$

$$\theta = \pi - \alpha - \beta = 0.760 \text{ rads}$$

6 $e^y = \sin x + 1$ P(-\pi, 0)

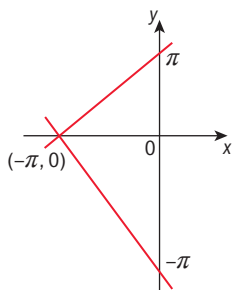
$$e^y \frac{dy}{dx} = -\cos x \quad \frac{dy}{dx} = \frac{-\cos x}{e^y}$$

$$\text{At P, } \frac{dy}{dx} = \frac{1}{e^0} = 1$$

$$\text{Tangent: } y = x + \pi$$

$$\text{Normal: } y = x - \pi$$

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times 2\pi \times \pi \\ &= \pi^2 \end{aligned}$$



Exercise 9F

1 a $f(x) = \tan x$ $f'(x) = \sec^2 x$

$$f''(x) = 2\sec x \sec x \tan x = 2\sec^2 x \tan x$$

$$f''\left(\frac{\pi}{3}\right) = 2(2)^2\sqrt{3} = 8\sqrt{3}$$

b $f(x) = x \sin x$, $f'(x) = \sin x + x \cos x$

$$f''(x) = \cos x + \cos x - x \sin x$$

$$= 2\cos x - x \sin x$$

$$f''(0) = 2$$

c $f(x) = (x^2 + 1) \cos x$

$$f'(x) = 2x \cos x - (x^2 + 1) \sin x$$

$$f''(x) = 2\cos x - 2x \sin x - [(x^2 + 1) \cos x + 2x \sin x]$$

$$= 2\cos x - 4x \sin x - (x^2 + 1) \cos x$$

$$f''(0) = 2 - 1 = 1$$

d $f(x) = \sqrt{x} \cos \frac{x}{2}$

$$f'(x) = -\frac{1}{2}\sqrt{x} \sin \frac{x}{2} + \frac{1}{2}x^{-\frac{1}{2}} \cos \frac{x}{2}$$

$$= \frac{-x \sin \frac{x}{2} + \cos \frac{x}{2}}{2\sqrt{x}}$$

$$f''(x) = \frac{2\sqrt{x} \left(-\sin \frac{x}{2} - \frac{x}{2} \cos \frac{x}{2} - \frac{1}{2} \sin \frac{x}{2} \right) - \left(-x \sin \frac{x}{2} + \cos \frac{x}{2} \right) x^{\frac{1}{2}}}{4x}$$

$$f''(1) = \frac{2 \left(-\frac{3}{2} \sin \frac{1}{2} - \frac{1}{2} \cos \frac{1}{2} \right) - \left(-\sin \frac{1}{2} + \cos \frac{1}{2} \right)}{4}$$

$$= \frac{-3 \sin \frac{1}{2} - \cos \frac{1}{2} + \sin \frac{1}{2} - \cos \frac{1}{2}}{4}$$

$$= \frac{1}{4} \left(-2 \sin \frac{1}{2} - 2 \cos \frac{1}{2} \right)$$

$$= -\frac{1}{2} \left(\sin \frac{1}{2} + \cos \frac{1}{2} \right)$$

e $f(x) = e^x \sin 2x$ $f'(x) = e^x (2\cos 2x + \sin 2x)$

$$f''(x) = e^x (-4\sin 2x + 2\cos 2x + 2\cos 2x + \sin 2x)$$

$$= e^x (4\cos 2x - 3\sin 2x)$$

$$f''\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}(0 - 3) = -3e^{\frac{\pi}{4}}$$

f $f(x) = 2x \sec x$, $f'(x) = 2\sec x + 2x \sec x \tan x$

$$= 2\sec x (1 + x \tan x)$$

$$f''(x) = 2\sec x (\tan x + x \sec^2 x)$$

$$+ 2\sec x \tan x (1 + x \tan x)$$

$$f''(\pi) = -2(0 + \pi) + 2(-1)(0) = -2\pi$$

2 a $f(x) = \cos x$ $f'(x) = -\sin x$

$$f''(x) = -\cos x \quad f^{(3)}(x) = \sin x$$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 4k - 3 \\ -\cos x, & n = 4k - 2 \\ \sin x, & n = 4k - 1 \\ \cos x, & n = 4k \end{cases} \quad k \in \mathbb{Z}^+$$

$$g^{(n)}(x) = \begin{cases} 3^n \cos 3x & n = 4k - 3 \\ -3^n \sin 3x & n = 4k - 2 \\ -3^n \cos 3x & n = 4k - 1 \\ 3^n \sin 3x & n = 4k \end{cases} \quad k \in \mathbb{Z}^+$$

$$h^{(n)}(x) = \begin{cases} -a^n \sin(ax+b) & n=4k-3 \\ -a^n \cos(ax+b) & n=4k-2 \\ a^n \sin(ax+b) & n=4k-1 \\ a^n \cos(ax+b) & n=4k \end{cases} \quad k \in \mathbb{Z}^+$$

$$\mathbf{a} \quad a_1 = f\left(\frac{\pi}{8}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x) = -4\sin 2x \quad a_3 = f''\left(\frac{\pi}{8}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$$

$$f^{(3)}(x) = -8\cos 2x \quad a_4 = f^{(3)}\left(\frac{\pi}{8}\right) = -\frac{8}{\sqrt{2}} = -4\sqrt{2}$$

$$\frac{1}{\sqrt{2}}, \sqrt{2}, -2\sqrt{2}, -4\sqrt{2}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{1}{\sqrt{2}} (1 + 2 - 4 - 8 + 16 + 32 - 64 \\ & - 128 + 256 + 512) \\ & = \frac{615}{\sqrt{2}} \text{ or } \frac{615\sqrt{2}}{2} \end{aligned}$$

$$P(0): f(x) = \sin x$$

Assume P(k): $f^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$

$$\begin{aligned} \text{Prove P}(k+1) \quad f^{(k+1)}(x) &= \cos\left(x + \frac{k\pi}{2}\right) \\ &= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin\left(x + (k+1)\frac{\pi}{2}\right) \end{aligned}$$

$\therefore$ by induction,

$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), n = 0, 1, 2, \dots$$

b $P(n): g(x) = \cos x \Rightarrow g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right),$
 $n = 0, 1, 2, \dots$

$$\text{P(0): } g(x) = \sin\left(x + \frac{\pi}{2}\right) = \cos x$$

Assume P(k): $g^{(k)}(x) = \sin\left(x + \frac{(k+1)\pi}{2}\right)$

Prove $P(k+1)$ $g^{(k+1)}(x) = \cos\left(x + \frac{(k+1)\pi}{2}\right)$

$$= \sin\left(x + \frac{(k+1)\pi}{2} + \frac{\pi}{2}\right)$$

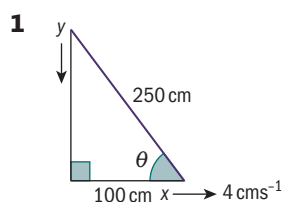
$$= \sin\left(x + \frac{(k+2)\pi}{2}\right)$$

$$\therefore P(k) \Rightarrow P(k + 1) \text{ and } P(0) \text{ is true}$$

$\therefore$ by induction,

$$g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right), \quad n = 0, 1, 2, \dots$$

Exercise 9G



$$\frac{dx}{dt} = 4 \quad \cos \theta = \frac{x}{250}$$

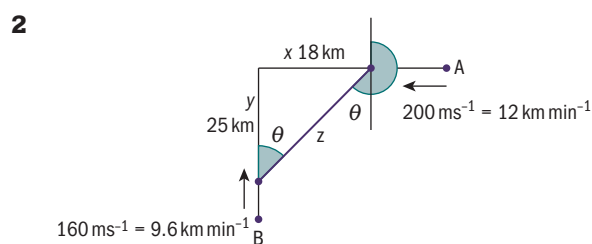
$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{250} \frac{dx}{dt}$$

$$x = 100 \Rightarrow \cos \theta = \frac{100}{250} = \frac{2}{5}$$

$$\sin \theta = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$$

$$-\frac{\sqrt{21}}{5} \frac{d\theta}{dt} = \frac{4}{250} \quad \therefore \quad \frac{d\theta}{dt} = -0.0175 \text{ cs}^{-1}$$

the angle is decreasing at a rate of 0.0175 cs^{-1}



a $x = 18 - 12t \Rightarrow \frac{dx}{dt} = -12$

$$y = 25 - 9.6t \Rightarrow \frac{dy}{dt} = -9.6$$

$$z^2 = x^2 + y^2 \quad \therefore 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$t = 0.5 \Rightarrow x = 12, y = 20.2,$$

$$z = \sqrt{552.04} = 23.4955\dots$$

$$23.4955 \frac{dz}{dt} = 12(-12) + 20.2(-9.6)$$

$$\therefore \frac{dz}{dt} = -14.4 \text{ km min}^{-1} \approx -240 \text{ ms}^{-1}$$

Approaching each other at 240 ms^{-1}

b bearing = $\pi + \theta$

$\therefore$ rate of change of bearing = $\frac{d\theta}{dt}$

$\tan \theta = \frac{x}{y} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \left(y \frac{dx}{dt} - x \frac{dy}{dt} \right) y^2$

$t = 1 \Rightarrow x = 6, y = 15.4, \tan \theta = \frac{6}{15.4}$

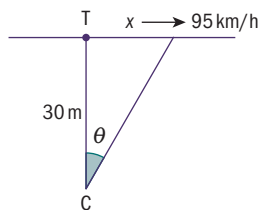
$\sec^2 \theta = 1 + \left(\frac{6}{15.4} \right)^2 = \frac{6829}{5929}$

$\frac{6829}{5929} \frac{d\theta}{dt} = \frac{15.4(-12) - 6(-9.6)}{15.4^2}$

$\frac{d\theta}{dt} = -0.466 \text{ c min}^{-1}$

bearing is decreasing at a rate of 0.466 c min^{-1}

3 a



$\frac{dx}{dt} = 95000 \text{ mh}^{-1}$

$\tan \theta = \frac{x}{30}$

$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$

$x = 0, \theta = 0 \Rightarrow \sec^2 \theta = 1 \Rightarrow \frac{d\theta}{dt} = \frac{95000}{30}$
 $= 31.667 \text{ ch}^{-1}$

$= 52.8 \text{ c min}^{-1} \text{ or } 0.880 \text{ cs}^{-1}$

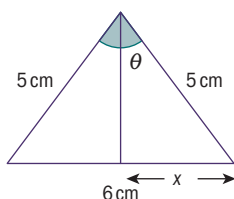
b After 1 sec, $x = \frac{95000}{3600} = 26.338 \dots \text{ m}$

$\tan \theta = \frac{26.338 \dots}{30} = 0.8796 \dots \quad \sec^2 \theta = 1.7737 \dots$

$1.7737 \dots \frac{d\theta}{dt} = \frac{95000}{30}$

$\therefore \frac{d\theta}{dt} = 1785.3 \text{ c/h} = 29.8 \text{ c min}^{-1} \text{ or } 0.496 \text{ cs}^{-1}$

4 a



$\frac{dx}{dt} = -0.05 \text{ cms}^{-1}$

$\frac{d}{dt}(2\theta) = 2 \frac{d\theta}{dt}$

$\sin \theta = \frac{x}{5}$

$\cos \theta \frac{d\theta}{dt} = \frac{1}{5} \frac{dx}{dt}$

At $t = 0, \cos \theta = \frac{4}{5}$

$\therefore \frac{d\theta}{dt} = \frac{1}{5} \times -0.05$

$\therefore \frac{d\theta}{dt} = -0.0125$

$\therefore$ angle is decreasing at a rate of 0.0125 cs^{-1}

b $\theta = 30^\circ$ when equilateral

$\therefore \cos 30^\circ \frac{d\theta}{dt} = \frac{1}{5} \frac{dx}{dt} = -0.01$

$\therefore \frac{d\theta}{dt} = -0.115$

$\therefore$ angle 2θ is decreasing at 0.0231 cs^{-1}

5 a $\frac{dv}{dt} = -2 \text{ cm}^3 \text{ min}^{-1}$

$v = \frac{4}{3} \pi r^3 \quad \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$r = 12, \quad -2 = 4\pi(12)^2 \frac{dr}{dt} \quad \frac{dr}{dt} = -\frac{1}{288\pi} \text{ cm min}^{-1}$
 $= -0.00111 \text{ cm min}^{-1}$

radius is decreasing at $0.00111 \text{ cm min}^{-1}$

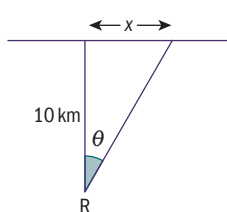
b $A = 4\pi r^2 \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

$r = 4, \quad -2 = 4\pi(4)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-1}{32\pi} \text{ cm min}^{-1}$

$\frac{dA}{dt} = 8\pi(4) \left(\frac{-1}{32\pi} \right) = -1 \text{ cm}^2 \text{ min}^{-1},$

decreasing at $1 \text{ cm}^2 \text{ min}^{-1},$

6 a



$\frac{dx}{dt} = 1025 \text{ km h}^{-1}$

$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$

$x = 8, \tan \theta = \frac{4}{5} \quad \sec^2 \theta = 1 + \frac{16}{25} = \frac{41}{25}$

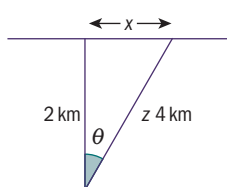
$\frac{41}{25} \frac{d\theta}{dt} = 102.5 \Rightarrow \frac{d\theta}{dt} = 62.5 \text{ ch}^{-1}$

$\frac{d\theta}{dt} = 0.01761 \text{ cs}^{-1}$
 $= 0.995 \text{ degs s}^{-1}$

b $x = 0, \theta = 0, \sec^2 \theta = 1$

$\frac{d\theta}{dt} = 102.5 \text{ ch}^{-1} = 0.028472 \dots \text{ cs}^{-1} = 1.63 \text{ degs s}^{-1}$

7 a



$\frac{dx}{dt} = 75 \text{ km h}^{-1}$

$z^2 = 4 + x^2$

$2z \frac{dz}{dt} = 2x \frac{dx}{dt}$

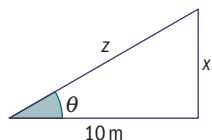
when $z = 4, 16 = 4 + x^2 \quad \therefore x = \sqrt{12}$

$4 \frac{dz}{dt} = \sqrt{12} (75)$

$\frac{dz}{dt} = \frac{75\sqrt{3}}{2} = 65.0 \text{ km h}^{-1}$

b $\tan \theta = \frac{x}{2}$ $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{2} \frac{dx}{dt}$
 $x = \sqrt{12}$, $\tan \theta = \sqrt{3}$, $\sec^2 \theta = 1 + (\sqrt{3})^2 = 4$
 $4 \frac{d\theta}{dt} = \frac{75}{2}$
 $\frac{d\theta}{dt} = \frac{75}{8} = 9.375 \text{ c/h} = 0.00260 \text{ c sec}^{-1}$
 $= 0.1 \text{ deg s}^{-1}$ (nearest tenth)

8 a



$$\frac{dz}{dt} = 5 \text{ ms}^{-1}$$

$$\cos \theta = \frac{10}{z}$$

$$-\sin \theta \frac{d\theta}{dt} = \frac{-10}{z^2} \frac{dz}{dt}$$

$$z = 20, \cos \theta = \frac{10}{20} = \frac{1}{2}, \sin^2 \theta = 1 - \frac{1}{4} = \frac{3}{4},$$

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\frac{\sqrt{3}}{2} \frac{d\theta}{dt} = \frac{10}{20^2} (5) \Rightarrow \frac{d\theta}{dt} = \frac{1}{4\sqrt{3}} = 0.144 \text{ cs}^{-1}$$

$$\text{or } 8.27 \text{ deg s}^{-1}$$

b $z^2 = 100 + x^2$

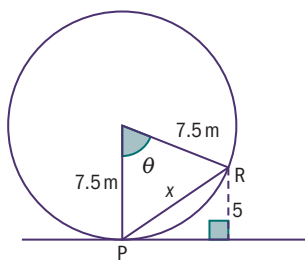
$$2z \frac{dz}{dt} = 2x \frac{dx}{dt}$$

$$z = 20, x^2 = 300 \Rightarrow x = 10\sqrt{3}$$

$$20(5) = 10\sqrt{3} \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{10}{\sqrt{3}} = 5.77 \text{ ms}^{-1}$$

9



$$\frac{d\theta}{dt} = \frac{4\pi}{60} = \frac{\pi}{15} \text{ cs}^{-1}$$

$$x^2 = 7.5^2 + 7.5^2 - 2 \times 7.5^2 \times \cos \theta$$

$$x^2 = 112.5 - 112.5 \cos \theta$$

$$2x \frac{dx}{dt} = 112.5 \sin \theta \frac{d\theta}{dt}$$

When height is 5 m, $\cos \theta = \frac{2.5}{7.5} = \frac{1}{3}$,

$$\sin^2 \theta = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\therefore \sin \theta = \frac{2\sqrt{2}}{3}$$

$$x^2 = 112.5 - 112.5 \left(\frac{1}{3} \right) = 75 \quad \therefore x = 5\sqrt{3}$$

$$10\sqrt{3} \frac{dx}{dt} = 112.5 \left(\frac{2\sqrt{2}}{3} \right) \left(\frac{\pi}{15} \right)$$

$$\therefore \frac{dx}{dt} = \frac{\pi\sqrt{2}}{2\sqrt{3}} = 1.28 \text{ ms}^{-1}$$

Exercise 9H

- 1 a** $\int \sin 3x \, dx = -\frac{1}{3} \cos 3x + c$
- b** $\int \cos(2x+1) \, dx = \frac{1}{2} \sin(2x+1) + c$
- c** $\int \sec^2 3x \, dx = \frac{1}{3} \tan 3x + c$
- d** $\int \sec^2(1-x) \, dx = -\tan(1-x) + c$
- e** $\int \sin\left(\frac{5x-1}{3}\right) dx = -\frac{3}{5} \cos\left(\frac{5x-1}{3}\right) + c$
- f** $\int \cos\left(\frac{3x+2}{7}\right) dx = \frac{7}{3} \sin\left(\frac{3x+2}{7}\right) + c$
- 2 a** $\int (1 - 2\cos^2 x) \, dx = \int -\cos 2x \, dx = -\frac{1}{2} \sin 2x + c$
- b** $\int (1 + \tan^2 x) \, dx = \int \sec^2 x \, dx = \tan x + c$
- c** $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$
 $\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx$
 $= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + c$
 $= \frac{1}{2} x - \frac{1}{4} \sin 2x + c$
- d** $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)$
 $\therefore \int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx$
 $= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + c$
 $= \frac{1}{2} x + \frac{1}{4} \sin 2x + c$
- e** $\int (1 - 2\sin^2(2x)) \, dx = \int \cos 4x \, dx = \frac{1}{4} \sin 4x + c$
- f** $\int (2 + 2\tan^2(5x)) \, dx = \int 2 \sec^2(5x) \, dx$
 $= \frac{2}{5} \tan 5x + c$
- g** $\int (1 + \tan^2 x)(1 - \sin^2 x) \, dx = \int \sec^2 x (\cos^2 x) \, dx$
 $= \int 1 \, dx = x + c$
- h** $\int 4 \sin^2 x \cos^2 x \, dx = \int \sin^2(2x) \, dx$
 $= \frac{1}{2} \int (1 - \cos 4x) \, dx$
 $= \frac{1}{2} \left(x - \frac{1}{4} \sin 4x \right) + c = \frac{1}{2} x - \frac{1}{8} \sin 4x + c$

Exercise 9I

- 1 a** $\int (2\sin x - 3\cos x) \, dx = -2\cos x - 3\sin x + c$
- b** $\int (x^2 - 7\sin x) \, dx = \frac{1}{3} x^3 + 7\cos x + c$

$$\begin{aligned} \text{c} \quad & \int \left(4e^x - \frac{1}{3} \sec^2 x \right) dx = 4e^x - \frac{1}{3} \tan x + c \\ \text{d} \quad & \int (1 - \sqrt{2x} + 7 \sin 3x) dx = x - \sqrt{2} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{7}{3} \cos 3x + c \\ & = x - \frac{2\sqrt{2}x^{\frac{3}{2}}}{3} - \frac{7}{3} \cos 3x + c \\ \text{e} \quad & \int \left(\frac{5}{2x} + \sec^2 \left(\frac{x}{3} \right) \right) dx = \frac{5}{2} \ln |x| + 3 \tan \frac{x}{3} + c \\ \text{f} \quad & \int \left(\frac{x}{x+1} - \sin \left(\frac{3x}{4} \right) \right) dx = \int \left(1 - \frac{1}{x+1} - \sin \left(\frac{3x}{4} \right) \right) dx \\ & = x - \ln |x+1| + \frac{4}{3} \cos \frac{3x}{4} + c \\ \text{g} \quad & \int \left(2^x + 5 \sin \frac{x}{2} - \cos \frac{2x}{3} \right) dx \\ & = \frac{2^x}{\ln 2} - 10 \cos \frac{x}{2} - \frac{3}{2} \sin \frac{2x}{3} + c \\ \text{h} \quad & \int (3^{-2x} - 11 \sec^2 (11x)) dx \\ & = \frac{-3^{-2x}}{2 \ln 3} - \tan (11x) + c \end{aligned}$$

Exercise 9J

$$\begin{aligned} \text{1 a} \quad & f'(x) = 5 - 2 \cos x \quad f(0) = 0 \\ & f(x) = 5x - 2 \sin x + c \\ & 0 = c \quad \therefore f(x) = 5x - 2 \sin x \\ \text{b} \quad & f'(x) = 4x - 6 \sin 2x \quad f(0) = 1 \\ & f(x) = 2x^2 + 3 \cos 2x + c \\ & 1 = 3 + c \quad \therefore c = -2 \quad f(x) = 2x^2 + 3 \cos 2x - 2 \\ \text{c} \quad & f'(x) = 3 \cos x - 2 \sec^2 x \quad f\left(\frac{\pi}{6}\right) = \frac{-2\sqrt{3}}{3} \\ & f(x) = 3 \sin x - 2 \tan x + c \\ & \frac{-2\sqrt{3}}{3} = \frac{3}{2} - \frac{2\sqrt{3}}{3} + c \quad \therefore c = \frac{-3}{2} \\ & f(x) = 3 \sin x - 2 \tan x - \frac{3}{2} \\ \text{d} \quad & f'(x) = 3x^2 - 2e^x + \cos 4x \quad f(0) = -5 \\ & f(x) = x^3 - 2e^x + \frac{1}{4} \sin 4x + c \\ & -5 = -2 + c \quad \therefore c = -3 \\ & f(x) = x^3 - 2e^x + \frac{1}{4} \sin 4x - 3 \\ \text{e} \quad & f'(x) = \frac{3}{x} + \cos(3x) - 4 \quad f(1) = \frac{\sin 3}{3} \\ & f(x) = 3 \ln |x| + \frac{1}{3} \sin(3x) - 4x + c \\ & \frac{\sin 3}{3} = \left(\frac{1}{3} \right) \sin 3 - 4 + c \quad \therefore c = 4 \\ & f(x) = 3 \ln |x| + \frac{1}{3} \sin(3x) - 4x + 4 \\ \text{f} \quad & f'(x) = \frac{7}{3-4x} - 8x + 4e^{2x-1} \quad f\left(\frac{1}{2}\right) = -1 \\ & f(x) = \frac{-7}{4} \ln |3-4x| - 4x^2 + 2e^{2x-1} + c \end{aligned}$$

$$\begin{aligned} -1 &= -1 + 2 + c \quad \therefore c = -2 \\ f(x) &= \frac{-7}{4} \ln |3-4x| - 4x^2 + 2e^{2x-1} - 2 \end{aligned}$$

$$\begin{aligned} \text{2 a} \quad & f''(x) = 4 \sin x \quad f'\left(\frac{\pi}{3}\right) = 0, \quad f(0) = 1 \\ & f'(x) = -4 \cos x + c_1 \\ & 0 = -2 + c_1 \quad \therefore c_1 = 2 \\ & f'(x) = -4 \cos x + 2 \\ & f(x) = -4 \sin x + 2x + c_2 \\ & 1 = c_2 \quad \therefore f(x) = -4 \sin x + 2x + 1 \\ \text{b} \quad & f''(x) = 1 + \cos x \quad f'(0) = 3, \quad f(1) = -\cos(1) \\ & f'(x) = x + \sin x + c_1 \\ & 3 = c_1 \quad f'(x) = x + \sin x + 3 \\ & f(x) = \frac{x^2}{2} - \cos x + 3x + c_2 \\ & -\cos(1) = \frac{1}{2} - \cos(1) + 3 + c_2, \quad c = -\frac{7}{2} \\ & f(x) = \frac{1}{2} x^2 - \cos x + 3x - \frac{7}{2} \\ \text{c} \quad & f''(x) = e^{1-x} + \sin(1-x), \quad f'(1) = 2, \quad f(1) = 2 \\ & f'(x) = -e^{1-x} + \cos(1-x) + c_1 \\ & 2 = -1 + 1 + c_1 \quad \therefore c_1 = 2 \\ & f'(x) = -e^{1-x} + \cos(1-x) + 2 \\ & f(x) = e^{1-x} - \sin(1-x) + 2x + c_2 \\ & 2 = 1 + 2 + c_2 \quad \therefore c_2 = -1 \\ & f(x) = e^{1-x} - \sin(1-x) + 2x - 1 \\ \text{d} \quad & f''(x) = e^{2x} + \sin(2x) + x^3 - 2x + 1, \quad f'(0) = 2, \quad f(0) = 2 \\ & f'(x) = \frac{1}{2} e^{2x} - \frac{1}{2} \cos(2x) + \frac{1}{4} x^4 - x^2 + x + c_1 \\ & 2 = \frac{1}{2} - \frac{1}{2} + c_1 \quad \therefore c_1 = 2 \\ & f'(x) = \frac{1}{2} e^{2x} - \frac{1}{2} \cos(2x) + \frac{1}{4} x^4 - x^2 + x + 2 \\ & f(x) = \frac{1}{4} e^{2x} - \frac{1}{4} \sin 2x + \frac{1}{20} x^5 - \frac{1}{3} x^3 + \frac{1}{2} x^2 + 2x + c_2 \\ & 2 = \frac{1}{4} + c_2 \quad \therefore c_2 = \frac{7}{4} \\ & f(x) = \frac{1}{4} e^{2x} - \frac{1}{4} \sin 2x + \frac{1}{20} x^5 - \frac{1}{3} x^3 + \frac{1}{2} x^2 + 2x + \frac{7}{4} \end{aligned}$$

Exercise 9K

$$\begin{aligned} \text{1 a} \quad & \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} (2x - \sin x) dx = \left[x^2 + \cos x \right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \\ & = \left(\frac{\pi^2}{4} + 0 \right) - \left(\frac{\pi^2}{9} + \frac{1}{2} \right) = \frac{5\pi^2}{36} - \frac{1}{2} \\ \text{b} \quad & \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (5 + \cos x) dx = \left[5x + \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ & = \left(\frac{5\pi}{2} + 1 \right) - \left(\frac{5\pi}{6} + \frac{1}{2} \right) = \frac{5\pi}{3} + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{c} \quad \int_0^{\frac{\pi}{4}} (2\sec^2 x + 1) dx &= [2\tan x + x]_0^{\frac{\pi}{4}} \\ &= \left(2 + \frac{\pi}{4}\right) - (0) = 2 + \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{d} \quad \int_0^{\frac{\pi}{3}} (e^x + 2\sin x) dx &= [e^x - 2\cos x]_0^{\frac{\pi}{3}} \\ &= (e^{\frac{\pi}{3}} - 1) - (1 - 2) = e^{\frac{\pi}{3}} \end{aligned}$$

$$\begin{aligned} \text{e} \quad \int_{-2\pi}^{2\pi} \left(3^{-x} + \frac{1}{4}\cos\frac{x}{4}\right) dx &= \left[\frac{-3^{-x}}{\ln 3} + \sin\frac{x}{4}\right]_{-2\pi}^{2\pi} \\ &= \left(\frac{-3^{-2\pi}}{\ln 3} + 1\right) - \left(\frac{-3^{2\pi}}{\ln 3} - 1\right) = \frac{3^{2\pi} - 3^{-2\pi}}{\ln 3} + 2 \end{aligned}$$

$$\begin{aligned} \text{f} \quad \int_0^{\frac{\pi}{2}} \left(\frac{e^{3x}}{3} - \frac{2\sin 2x}{5}\right) dx &= \left[\frac{e^{3x}}{9} + \frac{1}{5}\cos 2x\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{1}{9}e^{\frac{3\pi}{2}} - \frac{1}{5}\right) - \left(\frac{1}{9} + \frac{1}{5}\right) = \frac{1}{9}(e^{\frac{3\pi}{2}} - 1) - \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{g} \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1 - \frac{x}{2} + 2\sin 2x\right) dx &= \left[x - \frac{x^2}{4} - \cos 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) - \left(-\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \text{h} \quad \int_0^{\frac{\pi}{2}} (2^x + 3\cos 6x) dx &= \left[\frac{2^x}{\ln 2} + \frac{1}{2}\sin 6x\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{2^{\frac{\pi}{2}}}{\ln 2} + \frac{1}{2}\right) - \left(\frac{1}{\ln 2} + 0\right) = \frac{2^{\frac{\pi}{2}} - 1}{\ln 2} + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{i} \quad \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (x^2 + 2\sec^2 2x) dx &= \left[\frac{x^3}{3} + \tan 2x\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \\ &= \left(\frac{\pi^3}{1536} + 1\right) - \left(-\frac{\pi^3}{1536} - 1\right) = \frac{\pi^3}{768} + 2 \end{aligned}$$

$$\begin{aligned} \text{j} \quad \int_0^{\pi} (16e^{8x} + 9\sin 3x) dx &= [2e^{8x} - 3\cos 3x]_0^{\pi} \\ &= (2e^{8\pi} + 3) - (2 - 3) = 2e^{8\pi} + 4 \end{aligned}$$

Exercise 9L

- 1 a $\int 2x \sin x^2 dx = -\cos x^2 + c$
- b $\int 3x^2 \sqrt{x^3 + 3} dx = \frac{2}{3}(x^3 + 3)^{\frac{3}{2}} + c$
- c $\int (3 - 4x)e^{1+3x-2x^2} dx = e^{1+3x-2x^2} + c$
- d $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$
 $= -\ln|\cos x| + c$ or $\ln|\sec x| + c$
- e $\int 2\cos 2x e^{\sin 2x} dx = e^{\sin 2x} + c$

$$\text{f} \quad \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = e^{\sqrt{x}} + c$$

$$\text{g} \quad \int 2^x \ln 2 \sin(2^x) dx = -\cos(2^x) + c$$

$$\text{h} \quad \int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \frac{1}{2}(\arcsin x)^2 + c$$

$$\text{i} \quad \int \frac{2 \arctan 2x}{1-4x^2} dx = \frac{1}{2}(\arctan 2x)^2 + c$$

$$2 \text{ a} \quad \int x \cos x^2 dx = \frac{1}{2} \sin x^2 + c$$

$$\begin{aligned} \text{b} \quad \int x^5 \sqrt[3]{x^6 - 1} dx &= \frac{1}{6}(x^6 - 1)^{\frac{4}{3}} \cdot \frac{3}{4} + c \\ &= \frac{1}{8}(x^6 - 1)^{\frac{4}{3}} + c \end{aligned}$$

$$\text{c} \quad \int (x+2)e^{3x^2+12x-7} dx = \frac{1}{6}e^{3x^2+12x-7} + c$$

$$\begin{aligned} \text{d} \quad \int \frac{\tan(5x+4)}{5} dx &= \int \frac{\sin(5x+4)}{5\cos(5x+4)} dx \\ &= \frac{-1}{25} \ln|\cos(5x+4)| + c \text{ or } \frac{1}{25} \ln|\sec(5x+4)| + c \end{aligned}$$

$$\text{e} \quad \int \sin 3x \cdot 3^{\cos 3x} dx = \frac{-3^{\cos 3x}}{3 \ln 3} + c$$

$$\text{f} \quad \int \frac{\sin \sqrt[4]{x}}{\sqrt[4]{x^3}} dx = -4 \cos \sqrt[4]{x} + c$$

$$\text{g} \quad \int 5x \cos(5^x) dx = \frac{\sin(5^x)}{\ln 5} + c$$

$$\text{h} \quad \int \frac{e^{2x} + e^{-2x}}{e^{-2x} - e^{2x}} dx = -\frac{1}{2} \ln|e^{-2x} - e^{2x}| + c$$

$$\text{i} \quad \int \frac{\sqrt{\arctan \frac{x}{3}}}{9+x^2} dx = \frac{2}{9} \left(\arctan \frac{x}{3}\right)^{\frac{3}{2}} + c$$

$$\text{j} \quad \int (x^2 + x) \cos\left(x^3 + \frac{3}{2}x^2\right) dx = \frac{1}{3} \sin\left(x^3 + \frac{3}{2}x^2\right) + c$$

$$\text{k} \quad \int \frac{\arcsin^2(2x+1)}{\sqrt{-x-x^2}} dx = \frac{1}{3} \arcsin^3(2x+1) + c$$

Exercise 9M

$$1 \quad \int_0^1 3x^2(x^3 - 1)^4 dx = \left[\frac{1}{5}(x^3 - 1)^5\right]_0^1 = 0 - \left(-\frac{1}{5}\right) = \frac{1}{5}$$

$$2 \quad \int_0^3 \frac{2x}{x^2 + 1} dx = [\ln|x^2 + 1|]_0^3 = \ln 10 - \ln 1 = \ln 10$$

$$\begin{aligned} 3 \quad \int_0^{\frac{\pi}{6}} \cos x \sqrt{\sin x} dx &= \left[\frac{2}{3}(\sin x)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}} \\ &= \frac{2}{3} \left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6} \end{aligned}$$

$$4 \quad \int_1^{e^3} \frac{\ln x}{x} dx = \left[\frac{1}{2}(\ln x)^2\right]_1^{e^3} = \frac{1}{2}(\ln e^3)^2 - 0 = \frac{9}{2}$$

$$5 \quad \int_0^{\ln 2} \frac{e^x}{e^x + 1} dx = [\ln(e^x + 1)]_0^{\ln 2} = \ln(e^{\ln 2} + 1) - \ln 2 = \ln \frac{3}{2}$$

$$6 \quad \int_0^{\frac{\pi}{6}} 2 \tan 2x dx = [-\ln(\cos 2x)]_0^{\frac{\pi}{6}} = -\ln\left(\frac{1}{2}\right) = \ln 2$$

$$7 \quad \int_0^1 (x^2 + x) \cos\left(x^3 + \frac{3}{2}x^2\right) dx = \frac{1}{3} \left[\sin\left(x^3 + \frac{3}{2}x^2\right) \right]_0^1 \\ = \frac{1}{3} \sin \frac{5}{2}$$

$$8 \quad \int_0^3 2^x \sqrt{2x+1} dx = \frac{1}{\ln 2} \left[\frac{2}{3} (2^x + 1)^{\frac{3}{2}} \right]_0^3 \\ = \frac{2}{3 \ln 2} \left(9^{\frac{3}{2}} - 2^{\frac{3}{2}} \right) = \frac{2}{3 \ln 2} (27 - 2\sqrt{2})$$

Exercise 9N

$$1 \quad \int x e^x dx = x e^x - \int e^x dx \quad u = x \quad \frac{dv}{dx} = e^x$$

$$= x e^x - e^x + c \\ = e^x (x - 1) + c$$

$$2 \quad \int (2x + 9) \cos x dx \quad u = 2x + 9 \\ = (2x + 9) \sin x - \int 2 \sin x dx \quad \frac{dv}{dx} = \cos x \\ = (2x + 9) \sin x + 2 \cos x + c \quad \frac{du}{dx} = 2 \\ v = \sin x$$

$$3 \quad \int (2 - 5x) \sin x dx \quad u = 2 - 5x \\ = -(2 - 5x) \cos x - \int 5 \cos x dx \quad \frac{dv}{dx} = \sin x \\ = (5x - 2) \cos x - 5 \sin x + c \quad \frac{du}{dx} = -5 \\ v = -\cos x$$

$$4 \quad \int (3x - 1) e^{3x} dx \quad u = 3x - 1 \\ = (3x - 1) \frac{1}{3} e^{3x} - \int e^{3x} dx \quad \frac{dv}{dx} = e^{3x} \\ = \frac{1}{3} (3x - 1) e^{3x} - \frac{1}{3} e^{3x} + c \quad \frac{du}{dx} = 3 \\ v = \frac{1}{3} e^{3x} \\ = \frac{1}{3} e^{3x} (3x - 2) + c$$

$$5 \quad \int (4x - 7) e^{4x-1} dx \quad u = 4x - 7 \\ = \frac{1}{4} e^{4x-1} (4x - 7) - \int e^{4x-1} dx \quad \frac{dv}{dx} = e^{4x-1} \\ = \frac{1}{4} e^{4x-1} (4x - 7) - \frac{1}{4} e^{4x-1} + c \quad \frac{du}{dx} = 4 \\ = \frac{1}{4} e^{4x-1} (4x - 8) + c \quad v = \frac{1}{4} e^{4x-1} \\ = e^{4x-1} (x - 2) + c$$

$$6 \quad \int \frac{x+3}{2} \sin(2x+3) dx \quad u = \frac{x+3}{2} \\ = -\frac{1}{4} (x+3) \cos(2x+3) \quad \frac{dv}{dx} = \sin(2x+3) \\ + \int \frac{1}{4} \cos(2x+3) dx \quad \frac{du}{dx} = \frac{1}{2} \\ v = -\frac{1}{2} \cos(2x+3)$$

$$= -\frac{1}{4} (x+3) \cos(2x+3) + \frac{1}{8} \sin(2x+3) + c$$

$$7 \quad \int \frac{3-x}{4} \cos\left(\frac{x}{4}\right) dx \quad u = \frac{3-x}{4} \\ = (3-x) \sin\left(\frac{x}{4}\right) + \int \sin\left(\frac{x}{4}\right) dx \quad \frac{dv}{dx} = \cos\left(\frac{x}{4}\right) \\ = (3-x) \sin\left(\frac{x}{4}\right) - 4 \cos\left(\frac{x}{4}\right) + c \quad \frac{du}{dx} = \frac{-1}{4} \\ v = 4 \sin\left(\frac{x}{4}\right)$$

$$8 \quad \int x 2^x dx \quad u = x \quad \frac{dv}{dx} = 2^x \\ = \frac{2^x x}{\ln 2} - \int \frac{2^x}{\ln 2} dx \quad \frac{du}{dx} = 1 \quad v = \frac{2^x}{\ln 2} \\ = \frac{2^x x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + c \\ = \frac{2^x (x \ln 2 - 1)}{(\ln 2)^2} + c$$

$$9 \quad \int (1-x) 5^x dx \quad u = 1-x \\ = \frac{(1-x) 5^x}{\ln 5} + \int \frac{5^x}{\ln 5} dx \quad \frac{dv}{dx} = 5^x \\ = \frac{(1-x) 5^x}{\ln 5} + \frac{5^x}{(\ln 5)^2} + c \quad \frac{du}{dx} = -1 \\ = \frac{5^x ((1-x) \ln 5 + 1)}{(\ln 5)^2} + c \quad v = \frac{5^x}{\ln 5}$$

$$10 \quad \int \frac{(2-x)}{7 \cdot 3^x} dx = \frac{1}{7} \int (2-x) 3^{-x} dx \quad u = 2-x \\ = \frac{1}{7} \left[\frac{-3^{-x} (2-x)}{\ln 3} - \int \frac{3^{-x}}{\ln 3} dx \right] \quad \frac{dv}{dx} = 3^{-x} \\ = \frac{1}{7} \left[\frac{3^{-x} (x-2)}{\ln 3} + \frac{3^{-x}}{(\ln 3)^2} \right] + c \quad \frac{du}{dx} = -1 \\ = \frac{3^{-x} ((x-2) \ln 3 + 1)}{7(\ln 3)^2} + c \quad v = \frac{-3^{-x}}{\ln 3}$$

$$11 \quad \int \frac{4x \cdot 3^x}{5^x} dx = \int 4x \left(\frac{3}{5}\right)^x dx \quad u = 4x \quad \frac{dv}{dx} = \left(\frac{3}{5}\right)^x \\ = 4x \left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)} - \int 4 \left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)} dx \quad \frac{du}{dx} = 4 \\ v = \left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)}$$

$$= \frac{4x \cdot 3^x}{5^x \ln\left(\frac{3}{5}\right)} - 4\left(\frac{3}{5}\right)^x \frac{1}{\left(\ln\left(\frac{3}{5}\right)\right)^2} + c$$

$$= \frac{4 \cdot 3^x \left(x \ln\left(\frac{3}{5}\right) - 1\right)}{5^x \left(\ln\left(\frac{3}{5}\right)\right)^2} + c$$

Exercise 90

1 $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

2 $\int (3x+2) \ln x \, dx$

$$= \left(\frac{3x^2}{2} + 2x\right) \ln x - \int \left(\frac{3x}{2} + 2\right) dx$$

$$= \left(\frac{3x^2}{2} + 2x\right) \ln x - \frac{3x^2}{4} - 2x + c$$

3 $\int (1-x) \ln x \, dx$

$$= \left(x - \frac{x^2}{2}\right) \ln x - \int \left(1 - \frac{x}{2}\right) dx$$

$$= \left(x - \frac{x^2}{2}\right) \ln x - x + \frac{x^2}{4} + c$$

4 $\int x \ln(4x) \, dx$

$$= \frac{x^2}{2} \ln(4x) - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \ln(4x) - \frac{x^2}{4} + c$$

$$= \frac{x^2}{4} (2 \ln(4x) - 1) + c$$

5 $\int (3x-2) \ln\left(\frac{x}{5}\right) dx$

$$= \left(\frac{3x^2}{2} - 2x\right) \ln\left(\frac{x}{5}\right) - \int \left(\frac{3x}{2} - 2\right) dx$$

$$= \left(\frac{3x^2}{2} - 2x\right) \ln\left(\frac{x}{5}\right) - \frac{3x^2}{4} + 2x + c$$

$$u = \ln x$$

$$\frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^2}{2}$$

$$u = \ln x$$

$$\frac{dv}{dx} = 3x + 2$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{3x^2}{2} + 2x$$

$$u = \ln x$$

$$\frac{dv}{dx} = 1 - x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = x - \frac{x^2}{2}$$

$$u = \ln(4x)$$

$$\frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{x^2}{2}$$

$$u = \ln\left(\frac{x}{5}\right)$$

$$\frac{dv}{dx} = 3x - 2$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{3x^2}{2} - 2x$$

6 $\int (3+4x) \ln(3+4x) dx$

$$\text{Let } u = 3 + 4x, \quad \frac{du}{dx} = 4$$

$$\int (3+4x) \ln(3+4x) dx = \int u \ln u \cdot \frac{1}{4} du$$

$$= \frac{1}{4} \left[\frac{u^2}{2} \ln u - \frac{u^2}{4} \right] + c \quad (\text{using result from qn1})$$

$$= \frac{u^2}{16} (2 \ln u - 1) + c$$

$$= \frac{(3+4x)^2}{16} (2 \ln(3+4x) - 1) + c$$

7 Let $t = 4 - 11x$, $\frac{dt}{dx} = -11$

$$x = \frac{1}{11}(4-t)$$

$$\therefore 5+7x = 5 + \frac{7}{11}(4-t) = \frac{83}{11} - \frac{7t}{11}$$

$$\int (5+7x) \ln(4-11x) dx = \frac{-1}{11}$$

$$\int \frac{(83-7t)}{11} \ln t \, dt$$

$$= \frac{1}{121} \int (7t-83) \ln t \cdot dt$$

$$= \frac{1}{121} \left[\left(\frac{7t^2}{2} - 83t \right) \ln t \right.$$

$$\left. - \int \left(\frac{7t}{2} - 83 \right) dt \right]$$

$$= \frac{1}{121} \left[\left(\frac{7t^2}{2} - 83t \right) \ln t - \frac{7t^2}{4} + 83t \right] + c$$

$$= \frac{1}{121} \left[\left(\frac{7}{2}(4-11x)^2 - 83(4-11x) \right) \ln(4-11x) \right.$$

$$\left. - \frac{7}{4}(4-11x)^2 + 83(4-11x) \right] + c$$

$$= \frac{1}{121} \left[\left(\frac{7}{2}(16-88x+121x^2) - 332+913x \right) \ln(4-11x) \right.$$

$$\left. - \frac{7}{4}(16-88x+121x^2) + 332-913x \right] + c$$

$$= \frac{1}{121} \left[\left(\frac{847}{2}x^2 + 605x - 276 \right) \ln(4-11x) \right.$$

$$\left. - \frac{847}{4}x^2 - 759x + 304 \right] + c$$

8 $\int x^2 \ln x \, dx$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + c$$

$$u = \ln x$$

$$\frac{dv}{dx} = x^2$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{x^3}{3}$$

$$\begin{aligned}
 9 \quad & \int (2 - x + x^2) \ln(3x) \, dx & u = \ln(3x) \\
 & & \frac{dv}{dx} = 2 - x + x^2 \\
 & = \left(2x - \frac{x^2}{2} + \frac{x^3}{3} \right) \ln(3x) - & \frac{du}{dx} = \frac{1}{x} \\
 & \int \left(2 - \frac{x}{2} + \frac{x^2}{3} \right) dx & v = 2x - \frac{x^2}{2} + \frac{x^3}{3} \\
 & = \left(2x - \frac{x^2}{2} + \frac{x^3}{3} \right) \ln(3x) - 2x + \frac{x^2}{4} - \frac{x^3}{9} + c
 \end{aligned}$$

Exercise 9P

$$\begin{aligned}
 1 \quad & \int \log x \, dx & u = \log x & \frac{dv}{dx} = 1 \\
 & = x \log x - \int \frac{1}{x \ln 10} dx & \frac{du}{dx} = \frac{1}{x \ln 10} & v = x \\
 & = x \log x - \frac{x}{\ln 10} + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad & \int \log_a x \, dx & u = \log_a x & \frac{dv}{dx} = 1 \\
 & = x \log_a x - \int \frac{1}{x \ln a} dx & \frac{du}{dx} = \frac{1}{x \ln a} & v = x \\
 & = x \log_a x - \frac{x}{\ln a} + c
 \end{aligned}$$

$$\begin{aligned}
 3 \quad & \int \arctan x \, dx & u = \arctan x & \frac{dv}{dx} = 1 \\
 & = x \arctan x - \int \frac{x}{1+x^2} dx & \frac{du}{dx} = \frac{1}{1+x^2} & v = x \\
 & = x \arctan x - \frac{1}{2} \ln |1+x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 4 \quad & \int \arccos x \, dx & u = \arccos x & \frac{dv}{dx} = 1 \\
 & = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx & \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} & v = x \\
 & = x \arccos x - \frac{1}{2} \cdot 2(1-x^2)^{\frac{1}{2}} + c \\
 & = x \arccos x - \sqrt{1-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad & \int 2x \arctan x \, dx & u = \arctan x & \frac{dv}{dx} = 2x \\
 & = x^2 \arctan x - \int \frac{x^2}{1+x^2} dx & \frac{du}{dx} = \frac{1}{1+x^2} & v = x^2 \\
 & = x^2 \arctan x - \int \left(1 - \frac{1}{1+x^2} \right) dx \\
 & = x^2 \arctan x - x + \arctan x + c \\
 & = (x^2 + 1) \arctan x - x + c
 \end{aligned}$$

$$\begin{aligned}
 6 \quad & \int x^2 \arcsin x \, dx & u = \arcsin x & \frac{dv}{dx} = x^2 \\
 & = \frac{x^3}{3} \arcsin x - \int \frac{x^3}{3\sqrt{1-x^2}} dx & \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} & v = \frac{x^3}{3}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Let } t = 1-x^2 \quad \frac{dt}{dx} = -2x \\
 & \therefore x \, dx = -\frac{1}{2} dt \\
 & x^2 = 1-t \\
 & \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx = \frac{1}{3} \int \frac{1-t}{\sqrt{1-t}} \left(-\frac{1}{2} \right) dt = -\frac{1}{6} \int (t^{\frac{-1}{2}} - t^{\frac{1}{2}}) dt \\
 & = -\frac{1}{6} \left(2t^{\frac{1}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right) + c = \frac{1}{3} \left(\frac{1}{3} t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) + c \\
 & = \frac{1}{9} t^{\frac{1}{2}} (t-3) + c = \frac{1}{9} \sqrt{1-x^2} (1-x^2-3) + c \\
 & = -\frac{1}{9} (x^2+2) \sqrt{1-x^2} + c \\
 & \therefore \int x^2 \arcsin x \, dx = \frac{x^3}{3} \arcsin x + \frac{1}{9} (x^2+2) \sqrt{1-x^2} - c
 \end{aligned}$$

Exercise 9Q

$$\begin{aligned}
 1 \quad & \int x^2 e^x dx & u = x^2 & \frac{dv}{dx} = e^x \\
 & = x^2 e^x - \int 2x e^x dx & \frac{dv}{dx} = 2x & v = e^x \\
 & = x^2 e^x - [2x e^x - \int 2e^x dx] & v = 2x & \frac{dv}{dx} = e^x \\
 & = x^2 e^x - 2x e^x + 2e^x + c & \frac{du}{dx} = 2 & v = e^x \\
 & = e^x (x^2 - 2x + 2) + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad & \int (x^2 + 1) \sin x \, dx & u = x^2 + 1 & \frac{dv}{dx} = \sin x \\
 & = -(x^2 + 1) \cos x + \int 2x \cos x \, dx & \frac{du}{dx} = 2x & v = -\cos x \\
 & = -(x^2 + 1) \cos x + 2x \sin x - & u = 2x & \frac{dv}{dx} = \cos x \\
 & \int 2 \sin x \, dx & \frac{du}{dx} = 2 & v = \sin x \\
 & = -(x^2 + 1) \cos x + 2x \sin x + & & \\
 & 2 \cos x + c & & \\
 & = (1 - x^2) \cos x + 2x \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 3 \quad & \int (2x - x^2) \cos x \, dx & u = 2x - x^2 & \frac{dv}{dx} = \cos x \\
 & = (2x - x^2) \sin x + \int (2x - 2) \sin x \, dx & \frac{du}{dx} = 2 - 2x & v = \sin x \\
 & = (2x - x^2) \sin x - (2x - 2) \cos x & u = 2x - 2 & \\
 & + \int 2 \cos x \, dx & & \\
 & = (2x - x^2) \sin x - (2x - 2) \cos x & \frac{du}{dx} = 2 & \\
 & + 2 \sin x + c & v = -\cos x & \\
 & = (2x - x^2 + 2) \sin x + & & \\
 & (2 - 2x) \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 4 \quad & \int (1+x-x^2)e^{2x} dx \\
 & u = 1+x-x^2 \\
 & \frac{dv}{dx} = e^{2x} \\
 & \frac{du}{dx} = 1-2x \\
 & v = \frac{1}{2}e^{2x} \\
 & u = 2x-1 \\
 & \frac{dv}{dx} = \frac{1}{2}e^{2x} \\
 & \frac{du}{dx} = 2 \\
 & v = \frac{1}{4}e^{2x} \\
 & = \frac{1}{2}(1+x-x^2)e^{2x} + \int (2x-1) \frac{1}{2} e^{2x} dx \\
 & = \frac{1}{2}(1+x-x^2)e^{2x} + \frac{1}{4}(2x-1)e^{2x} - \int \frac{1}{2} e^{2x} dx \\
 & = \frac{1}{2}(1+x-x^2)e^{2x} + \frac{1}{4}(2x-1)e^{2x} - \frac{1}{4}e^{2x} + c \\
 & = \frac{1}{2}e^x(1+x-x^2+x-\frac{1}{2}+\frac{1}{2}) + c \\
 & = \frac{1}{2}e^x(2x-x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad & \int (2x^2+x+3)\cos(2x) dx \\
 & u = 2x^2+x+3 \\
 & \frac{dv}{dx} = \cos(2x) \\
 & \frac{du}{dx} = 4x+1 \\
 & v = \frac{1}{2}\sin(2x) \\
 & u = 4x+1 \\
 & \frac{dv}{dx} = \sin 2x \\
 & \frac{du}{dx} = 4 \\
 & v = -\frac{1}{2}\cos 2x \\
 & = \frac{1}{2}(2x^2+x+3)\sin 2x - \frac{1}{2}\int (4x+1)\sin 2x dx \\
 & = \frac{1}{2}(2x^2+x+3)\sin 2x - \frac{1}{2}\left[\frac{-1}{2}(4x+1)\cos 2x + \int 2\cos 2x dx\right] \\
 & = \frac{1}{2}(2x^2+x+3)\sin 2x + \frac{1}{4}(4x+1)\cos 2x - \frac{1}{2}\sin 2x + c \\
 & = \frac{1}{2}(2x^2+x+2)\sin 2x + \frac{1}{4}(4x+1)\cos 2x + c
 \end{aligned}$$

$$\begin{aligned}
 6 \quad & \int x^2\sin(1-2x)dx \\
 & u = x^2 \\
 & \frac{dv}{dx} = \sin(1-2x) \\
 & \frac{du}{dx} = 2x \\
 & v = -\frac{1}{2}\cos(1-2x) \\
 & u = x \\
 & \frac{dv}{dx} = \cos(1-2x) \\
 & \frac{du}{dx} = 1 \\
 & v = -\frac{1}{2}\sin(1-2x) \\
 & = \frac{1}{2}x^2\cos(1-2x) - \int x\cos(1-2x) dx
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{2}x^2\cos(1-2x) - \left[-\frac{1}{2}x\sin(1-2x) + \int \frac{1}{2}\sin(1-2x)dx\right] \\
 & = \frac{1}{2}x^2\cos(1-2x) + \frac{1}{2}x\sin(1-2x) - \frac{1}{4}\cos(1-2x) + c \\
 & = \frac{1}{4}(2x^2-1)\cos(1-2x) + \frac{1}{2}x\sin(1-2x) + c
 \end{aligned}$$

$$\begin{aligned}
 7 \quad & \int x^2 3^x dx \\
 & u = x^2 \quad \frac{dv}{dx} = 3^x \\
 & \frac{du}{dx} = 2x \quad v = \frac{3^x}{\ln 3} \\
 & = \frac{x^2 3^x}{\ln 3} - \int 2x \cdot \frac{3^x}{\ln 3} dx \\
 & = \frac{x^2 3^x}{\ln 3} - \left[2x \cdot \frac{3^x}{(\ln 3)^2} - \int 2 \cdot \frac{3^x}{(\ln 3)^2} dx\right] \\
 & = \frac{x^2 3^x}{\ln 3} - 2x \cdot \frac{3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3} + c \\
 & = \frac{3^x}{(\ln 3)^3} [x^2 (\ln 3)^2 - 2x \ln 3 + 2] + c
 \end{aligned}$$

$$\begin{aligned}
 8 \quad & \int (1+x^3)e^{\frac{x}{2}} dx \\
 & u = 1+x^3 \quad \frac{dv}{dx} = e^{\frac{x}{2}} \\
 & \frac{du}{dx} = 3x^2 \quad v = 2e^{\frac{x}{2}} \\
 & u = 6x^2 \quad \frac{dv}{dx} = e^{\frac{x}{2}} \\
 & \frac{du}{dx} = 12x \quad v = 2e^{\frac{x}{2}} \\
 & = 2(1+x^3)e^{\frac{x}{2}} - \int 6x^2 e^{\frac{x}{2}} dx \\
 & = 2(1+x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + \int 24xe^{\frac{x}{2}} dx \\
 & = 2(1+x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - \int 48e^{\frac{x}{2}} dx \\
 & = 2(1+x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c \\
 & = e^{\frac{x}{2}} [2x^3 - 12x^2 + 48x - 94] + c
 \end{aligned}$$

$$\begin{aligned}
 9 \quad & \int (x^3+x^2)\sin 5x dx \\
 & u = (x^3+x^2) \\
 & \frac{dv}{dx} = \sin 5x \\
 & \frac{du}{dx} = 3x^2+2x \\
 & v = -\frac{1}{5}\cos 5x
 \end{aligned}$$

$$= \frac{-1}{5} (x^3 + x^2) \cos 5x + \int (3x^2 + 2x) \frac{1}{5} \cos 5x \, dx$$

$$= \frac{-1}{5} (x^3 + x^2) \cos 5x + \frac{1}{25} (3x^2 + 2x) \sin 5x - \int (6x + 2) \frac{1}{25} \sin 5x \, dx$$

$$= \frac{-1}{5} (x^3 + x^2) \cos 5x + \frac{1}{25} (3x^2 + 2x) \sin 5x + \frac{1}{125} (6x + 2) \cos 5x - \int \frac{6}{125} \cos 5x \, dx$$

$$= \frac{-1}{5} (x^3 + x^2) \cos 5x + \frac{1}{25} (3x^2 + 2x) \sin 5x + \frac{1}{125} (6x + 2) \cos 5x - \frac{6}{125} \sin 5x + c$$

$$= \frac{1}{125} (-25x^3 - 25x^2 + 6x + 2) \cos 5x + \frac{1}{625} (75x^2 + 50x - 6) \sin 5x + c$$

$$\mathbf{10} \int x^4 \cos x \, dx \quad u = x^4 \quad \frac{dv}{dx} = \cos x$$

$$= x^4 \sin x - \int 4x^3 \sin x \, dx \quad \frac{du}{dx} = 4x^3 \quad v = \sin x$$

$$= x^4 \sin x + 4x^3 \cos x - \int 12x^2 \cos x \, dx$$

$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x + \int 24x \sin x \, dx$$

$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + \int 24 \cos x \, dx$$

$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + c$$

$$= (x^4 - 12x^2 + 24) \sin x + (4x^3 - 24x) \cos x + c$$

$$\mathbf{11} \int x^5 e^{2x} \, dx \quad u = x^5 \quad \frac{dv}{dx} = e^{2x}$$

$$\frac{du}{dx} = 5x^4 \quad v = \frac{1}{2} e^{2x}$$

$$= \frac{1}{2} x^5 e^{2x} - \int \frac{5}{2} x^4 e^{2x} \, dx \quad u = \frac{5}{2} x^4 \quad \frac{dv}{dx} = e^{2x}$$

$$u = 3x^2 + 2x$$

$$\frac{dv}{dx} = \frac{1}{5} \cos 5x$$

$$\frac{du}{dx} = 6x + 2$$

$$v = \frac{1}{25} \sin 5x$$

$$u = 6x + 2$$

$$\frac{dv}{dx} = \frac{1}{25} \sin 5x$$

$$du = 6$$

$$v = \frac{1}{25} \cos 5x$$

$$\frac{du}{dx} = 10x^3 \quad v = \frac{1}{2} e^{2x}$$

$$= \frac{1}{2} x^5 e^{2x} - \frac{5}{4} x^4 e^{2x} + \int 5x^3 e^{2x} \, dx \quad u = 5x^3 \quad \frac{dv}{dx} = e^{2x}$$

$$\frac{du}{dx} = 15x^2 \quad v = \frac{1}{2} e^{2x}$$

$$= \frac{1}{2} x^5 e^{2x} - \frac{5}{4} x^4 e^{2x} + \frac{5}{2} x^3 e^{2x} - \int \frac{15}{2} x^2 e^{2x} \, dx \quad u = \frac{15}{2} x^2 \quad \frac{dv}{dx} = e^{2x}$$

$$\frac{du}{dx} = 15x \quad v = \frac{1}{2} e^{2x}$$

$$= \frac{1}{2} x^5 e^{2x} - \frac{5}{4} x^4 e^{2x} + \frac{5}{2} x^3 e^{2x} - \frac{15}{4} x^2 e^{2x} + \int \frac{15}{2} x e^{2x} \, dx \quad u = \frac{15}{2} x \quad \frac{dv}{dx} = e^{2x}$$

$$\frac{du}{dx} = \frac{15}{2} \quad v = \frac{1}{2} e^{2x}$$

$$= \frac{1}{2} x^5 e^{2x} - \frac{5}{4} x^4 e^{2x} + \frac{5}{2} x^3 e^{2x} - \frac{15}{4} x^2 e^{2x} + \frac{15}{4} x e^{2x} - \int \frac{15}{4} e^{2x} \, dx$$

$$= \frac{e^{2x}}{8} (4x^5 - 10x^4 + 20x^3 - 30x^2 + 30x - 15) + c$$

Exercise 9R

$$\mathbf{1} \int \sin x e^x \, dx \quad u = \sin x \quad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = \cos x \quad v = e^x$$

$$= e^x \sin x - \int e^x \cos x \, dx \quad u = \cos x \quad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = -\sin x \quad v = e^x$$

$$= e^x \sin x - e^x \cos x - \int \sin x e^x \, dx$$

$$2 \int \sin x e^x \, dx = e^x (\sin x - \cos x) + c$$

$$\int \sin x e^x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

$$\mathbf{2} \int e^{2x} \cos x \, dx \quad u = e^{2x} \quad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 2e^{2x} \quad v = \sin x$$

$$= e^{2x} \sin x - \int 2e^{2x} \sin x \, dx \quad u = 2e^{2x} \quad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = 4e^{2x} \quad v = -\cos x$$

$$= e^{2x} \sin x + 2e^{2x} \cos x - \int 4e^{2x} \cos x \, dx$$

$$5 \int e^{2x} \cos x \, dx = e^{2x} (\sin x + 2 \cos x)$$

$$\therefore \int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} (\sin x + 2 \cos x) + c$$

$$\begin{aligned}
 3 \quad \int \cos 3x e^{4x} dx & \quad u = \cos 3x \quad \frac{dv}{dx} = e^{4x} \\
 & \quad \frac{du}{dx} = -3 \sin 3x \quad v = \frac{1}{4} e^{4x} \\
 \int \cos 3x e^{4x} dx &= \frac{1}{4} e^{4x} \cos 3x - \int \frac{3}{4} \sin 3x e^{4x} dx \\
 &= \frac{1}{4} e^{4x} \cos 3x + \frac{3}{16} e^{4x} \sin 3x - \int \frac{9}{16} \cos 3x e^{4x} dx \\
 \frac{25}{16} \int \cos 3x e^{4x} dx &= \frac{e^{4x}}{16} (4 \cos 3x + 3 \sin 3x) \\
 \int \cos 3x e^{4x} dx &= \frac{1}{25} e^{4x} (4 \cos 3x + 3 \sin 3x) + c
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \int \frac{\sin(2x)}{e^x} dx &= \int \sin 2x \cdot e^{-x} dx \quad u = \sin 2x \quad \frac{dv}{dx} = e^{-x} \\
 & \quad \frac{du}{dx} = 2 \cos 2x \quad v = -e^{-x} \\
 &= -e^{-x} \sin 2x + \int 2 \cos 2x e^{-x} dx \quad u = 2 \cos 2x \quad \frac{dv}{dx} = e^{-x} \\
 & \quad \frac{du}{dx} = -4 \sin 2x \quad v = -e^{-x} \\
 &= -e^{-x} \sin 2x - 2e^{-x} \cos 2x - \int 4 \sin 2x e^{-x} dx \\
 5 \int \frac{\sin 2x}{e^x} dx &= -e^{-x} (\sin 2x + 2 \cos 2x) \\
 \therefore \int \frac{\sin 2x}{e^x} dx &= \frac{-1}{5} e^{-x} (\sin 2x + 2 \cos 2x) + c
 \end{aligned}$$

Exercise 9S

$$\begin{aligned}
 1 \quad \int x \sqrt{x+2} dx & \quad u = x+2 \quad dx = du \\
 &= \int (u-2) u^{\frac{1}{2}} du \\
 &= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right) du \\
 &= \frac{2}{5} u^{\frac{5}{2}} - \frac{4}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{15} u^{\frac{3}{2}} (3u-10) + c \\
 &= \frac{2}{15} (x+2)^{\frac{3}{2}} (3x-4) + c
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \int 3x \sqrt{1-2x} dx & \quad u = 1-2x \\
 & \quad x = \frac{1}{2} (1-u) \quad dx = -\frac{1}{2} du \\
 &= \int \frac{3}{2} (1-u) u^{\frac{1}{2}} \left(\frac{-1}{2} \right) du \\
 &= \frac{-3}{4} \int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du \\
 &= \frac{-3}{4} \left(\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right) + c
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{-3}{60} \left(10u^{\frac{3}{2}} - 6u^{\frac{5}{2}} \right) + c \\
 &= \frac{-u^{\frac{3}{2}}}{10} (5-3u) + c \\
 &= \frac{-(1-2x)^{\frac{3}{2}}}{10} (2+6x) + c = -\frac{1}{5} (1-2x)^{\frac{3}{2}} (1+3x) + c
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \int 5x^2 \sqrt{3+4x} dx & \quad u = 3+4x \\
 & \quad x = \frac{1}{4} (u-3) \quad dx = \frac{1}{4} du \\
 &= \int \frac{5}{16} (u^2 - 6u + 9) u^{\frac{1}{2}} du \\
 &= \frac{5}{64} \int \left(u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}} \right) du \\
 &= \frac{5}{64} \left(\frac{2}{7} u^{\frac{7}{2}} - \frac{12}{5} u^{\frac{5}{2}} + 6u^{\frac{3}{2}} \right) + c \\
 &= \frac{1}{448} \left(10u^{\frac{7}{2}} - 84u^{\frac{5}{2}} + 210u^{\frac{3}{2}} \right) + c \\
 &= \frac{u^{\frac{3}{2}}}{224} (5u^2 - 42u + 105) + c \\
 &= \frac{(3+4x)^{\frac{3}{2}}}{224} (5(9+24x+16x^2) - 42(3+4x) + 105) + c \\
 &= \frac{(3+4x)^{\frac{3}{2}}}{224} (24-48x+80x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \int x^3 \sqrt{x+3} dx & \\
 &= \int u = x+3 dx = du \\
 &= \int (u-3) u^{\frac{1}{2}} du (u^{\frac{4}{3}} - 3u^{\frac{1}{3}}) du \\
 &= \frac{3}{7} u^{\frac{7}{3}} - \frac{9}{4} u^{\frac{4}{3}} + c \\
 &= \frac{3u^{\frac{4}{3}}}{28} (4u-21) + c \\
 &= \frac{3}{28} (x+3)^{\frac{4}{3}} (4x-9) + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \int x^2 \sqrt[4]{x+1} dx & \quad u = x+1 \quad dx = du \\
 & \quad x = u-1 \\
 &= \int (u^2 - 2u + 1) u^{\frac{1}{4}} du = \int \left(u^{\frac{9}{4}} - 2u^{\frac{5}{4}} + u^{\frac{1}{4}} \right) du \\
 &= \frac{4}{13} u^{\frac{13}{4}} - \frac{8}{9} u^{\frac{9}{4}} + \frac{4}{5} u^{\frac{5}{4}} + c \\
 &= \frac{4u^{\frac{5}{4}}}{585} (45u^2 - 130u + 117) + c
 \end{aligned}$$

$$= \frac{4}{585} (x+1)^{\frac{5}{4}} (45u^2 - 130u + 117) + c$$

$$= \frac{4}{585} (x+1)^{\frac{5}{4}} (45x^2 - 40x + 32) + c$$

$$6 \quad \int x^3 \sqrt[5]{1-x} \, dx \quad \begin{array}{l} u = 1-x \quad dx = -du \\ x = 1-u \end{array}$$

$$= -\int (1-3u+3u^2-u^3) u^{\frac{1}{5}} \, du$$

$$= -\int (u^{\frac{1}{5}} - 3u^{\frac{6}{5}} + 3u^{\frac{11}{5}} - u^{\frac{16}{5}}) \, du$$

$$= -\frac{5}{6} u^{\frac{6}{5}} + \frac{15}{11} u^{\frac{11}{5}} - \frac{15}{16} u^{\frac{16}{5}} + \frac{5}{21} u^{\frac{21}{5}} + c$$

$$= -\frac{5}{3696} u^{\frac{6}{5}} (616 - 1008u + 693u^2 - 176u^3) + c$$

$$= -\frac{5}{3696} (1-x)^{\frac{6}{5}} (616 - 1008(1-x) + 693(1-2x+x^2) - 176(1-3x+3x^2-x^3)) + c$$

$$= -\frac{5(1-x)^{\frac{6}{5}}}{3696} (125 + 150x + 165x^2 + 176x^3) + c$$

Exercise 9T

$$1 \quad \int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (\cos x - \sin^2 x \cos x) \, dx$$

$$= \sin x - \frac{1}{3} \sin^3 x + c$$

$$2 \quad \int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx$$

$$= \int \frac{1 + 2\cos 2x + \cos^2 2x}{4} \, dx$$

$$= \int \left(\frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cdot \frac{1 + \cos 4x}{2} \right) \, dx$$

$$= \int \left(\frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) \, dx$$

$$= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$$

$$3 \quad \int \sin^5 \left(\frac{x}{5} \right) \, dx = \int \left(1 - \cos^2 \left(\frac{x}{5} \right) \right) \sin \left(\frac{x}{5} \right) \, dx$$

$$= \int \left(\sin \frac{x}{5} - 2\cos^2 \frac{x}{5} \sin \frac{x}{5} + \cos^4 \frac{x}{5} \sin \frac{x}{5} \right) \, dx$$

$$= -5\cos \frac{x}{5} + \frac{10}{3} \cos^3 \frac{x}{5} - \cos^5 \frac{x}{5} + c$$

$$4 \quad \int 48 \cos^6(2x) \, dx = \int 48 (\cos^2(2x))^3 \, dx$$

$$= \int 48 \left(\frac{1 + \cos 4x}{2} \right)^3 \, dx$$

$$= \int 6(1 + 3\cos 4x + 3\cos^2 4x + \cos^3 4x) \, dx$$

$$= \int (6 + 18\cos 4x + 9(1 + \cos 8x) + 6\cos 4x(1 - \sin^2 4x)) \, dx$$

$$= \int (15 + 24\cos 4x + 9\cos 8x - 6\cos 4x \sin^2 4x) \, dx$$

$$= 15x + 6\sin 4x + \frac{9}{8} \sin 8x - \frac{1}{2} \sin^3 4x + c$$

Exercise 9U

$$1 \quad \int \sqrt{4-x^2} \, dx \quad x = 2\sin \theta \quad dx = 2\cos \theta \, d\theta$$

$$4 - x^2 = 4 - 4\sin^2 \theta = 4(1 - \sin^2 \theta) = 4\cos^2 \theta$$

$$\sqrt{4-x^2} = 2\cos \theta$$

$$\int \sqrt{4-x^2} \, dx = \int 2\cos \theta \cdot 2\cos \theta \, d\theta = \int 4\cos^2 \theta \, d\theta$$

$$= \int 2(1 + \cos 2\theta) \, d\theta$$

$$= 2\theta + \sin 2\theta + c$$

$$= 2\theta + 2\sin \theta \cos \theta + c$$

$$\theta = \arcsin \frac{x}{2} \quad \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$$

$$\int \sqrt{4-x^2} \, dx = 2\arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2} + c$$

$$2 \quad \int \frac{1}{\sqrt{x^2-1}} \, dx \quad x = \sec \theta \quad dx = \sec \theta \tan \theta \, d\theta$$

$$x^2 - 1 = \sec^2 \theta - 1 = \tan^2 \theta$$

$$\sqrt{x^2-1} = \tan \theta$$

$$\int \frac{1}{\sqrt{x^2-1}} \, dx = \int \frac{1}{\tan \theta} \sec \theta \tan \theta \, d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \ln |\sec \theta + \tan \theta| + c$$

$$= \ln |x + \sqrt{x^2-1}| + c$$

$$3 \quad \int \sqrt{x^2+9} \, dx \quad x = 3\tan \theta \quad dx = 3\sec^2 \theta \, d\theta$$

$$x^2 + 9 = 9(\tan^2 \theta + 1) = 9\sec^2 \theta$$

$$\sqrt{x^2+9} = 3\sec \theta$$

$$\int \sqrt{x^2+9} \, dx = \int 3\sec \theta \cdot 3\sec^2 \theta \, d\theta$$

$$= \int 9\sec^3 \theta = 9 \int \sec \theta \sec^2 \theta \, d\theta$$

Using integration by parts:

$$u = \sec \theta \quad \frac{dv}{d\theta} = \sec^2 \theta$$

$$\frac{du}{d\theta} = \sec \theta \tan \theta \quad v = \tan \theta$$

$$\begin{aligned}
 \int \sec^3 \theta \, d\theta &= \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta \, d\theta \\
 &= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta \\
 &= \sec \theta \tan \theta - \int \sec^2 \theta \, d\theta + \int \sec \theta \, d\theta \\
 2 \int \sec^3 \theta \, d\theta &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \\
 \therefore \int \sec^3 \theta \, d\theta &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + c \\
 \therefore \int \sqrt{x^2 + 9} \, dx &= \frac{9}{2} \sec \theta \tan \theta + \frac{9}{2} \ln |\sec \theta \\
 &\quad + \tan \theta| + c
 \end{aligned}$$

$$\begin{aligned}
 \int \sqrt{x^2 + 9} \, dx &= \frac{1}{2} x \sqrt{x^2 + 9} + \frac{9}{2} \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| + c \\
 &= \frac{1}{2} x \sqrt{x^2 + 9} + \frac{9}{2} \ln |\sqrt{x^2 + 9} + x| + k
 \end{aligned}$$

$$\begin{aligned}
 4 \int \frac{3}{\sqrt{36 - x^2}} \, dx \quad x &= 6 \sin \theta \quad dx = 6 \cos \theta \, d\theta \\
 36 - x^2 &= 36 (1 - \sin^2 \theta) = 36 \cos^2 \theta \\
 \sqrt{36 - x^2} &= 6 \cos \theta \\
 \int \frac{3}{\sqrt{36 - x^2}} \, dx &= \int \frac{3}{6 \cos \theta} \cdot 6 \cos \theta \, d\theta = \int 3 \, d\theta \\
 &= 3\theta + c \\
 &= 3 \arcsin \frac{x}{6} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \int 3\sqrt{x^2 - 16} \, dx \quad x &= 4 \sec \theta \quad dx = 4 \sec \theta \tan \theta \, d\theta \\
 x^2 - 16 &= 16 (\sec^2 \theta - 1) = 16 \tan^2 \theta \\
 \sqrt{x^2 - 16} &= 4 \tan \theta \\
 \int 3\sqrt{x^2 - 16} \, dx &= \int 12 \tan \theta \cdot 4 \sec \theta \tan \theta \, d\theta \\
 &= \int 48 \sec \theta \tan^2 \theta \, d\theta \\
 &= \int 48 \sec \theta (\sec^2 \theta - 1) \, d\theta \\
 &= \int (48 \sec^3 \theta - 48 \sec \theta) \, d\theta \\
 &\quad (\text{see qn. 3 for } \int \sec^3 \theta) \\
 &= 48 \left(\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) \\
 &\quad - 48 \ln |\sec \theta + \tan \theta| + c \\
 &= 24 \sec \theta \tan \theta - 24 \ln |\sec \theta + \tan \theta| + c \\
 &= 6x \frac{1}{4} \sqrt{x^2 - 16} - 24 \ln \left| \frac{x}{4} + \frac{\sqrt{x^2 - 16}}{4} \right| + c \\
 &= \frac{3}{2} x \sqrt{x^2 - 16} - 24 \ln |x + \sqrt{x^2 - 16}| + k
 \end{aligned}$$

$$\begin{aligned}
 6 \int \frac{5}{\sqrt{x^2 + 121}} \, dx \quad x &= 11 \tan \theta \quad dx = 11 \sec^2 \theta \, d\theta \\
 x^2 + 121 &= 121 (\tan^2 \theta + 1) = 121 \sec^2 \theta \\
 \sqrt{x^2 + 121} &= 11 \sec \theta \\
 \int \frac{5}{\sqrt{x^2 + 121}} \, dx &= \int \frac{5}{11 \sec \theta} \cdot 11 \sec^2 \theta \, d\theta \\
 &= \int 5 \sec \theta \, d\theta \\
 &= 5 \ln |\sec \theta + \tan \theta| + c \\
 &= 5 \ln \left| \frac{\sqrt{x^2 + 121} + x}{11} \right| + c \\
 &= 5 \ln |\sqrt{x^2 + 121} + x| + k
 \end{aligned}$$

$$\begin{aligned}
 7 \int \frac{2}{\sqrt{81 - 4x^2}} \, dx \quad x &= \frac{9}{2} \sin \theta \quad dx = \frac{9}{2} \cos \theta \, d\theta \\
 81 - 4x^2 &= 81 - 81 \sin^2 \theta = 81 \cos^2 \theta \\
 \sqrt{81 - 4x^2} &= 9 \cos \theta
 \end{aligned}$$

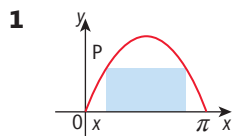
$$\begin{aligned}
 \int \frac{2}{\sqrt{81 - 4x^2}} \, dx &= \int \frac{2}{9 \cos \theta} \cdot \frac{9}{2} \cos \theta \, d\theta = \int 1 \, d\theta \\
 &= \theta + c \\
 &= \arcsin \frac{2x}{9} + c
 \end{aligned}$$

$$\begin{aligned}
 8 \int \sqrt{3x^2 - 75} \, dx &= \sqrt{3} \int \sqrt{x^2 - 25} \, dx \\
 x &= 5 \sec \theta \\
 dx &= 5 \sec \theta \tan \theta \, d\theta \\
 x^2 - 25 &= 25 (\sec^2 \theta - 1) = 25 \tan^2 \theta \\
 \int \sqrt{3x^2 - 75} \, dx &= \sqrt{3} \int 5 \tan \theta \cdot 5 \sec \theta \tan \theta \, d\theta \\
 &= \sqrt{3} \int 25 \sec \theta \tan^2 \theta \, d\theta \\
 &= \sqrt{3} \int 25 \sec \theta (\sec^2 \theta - 1) \, d\theta \\
 &\quad (\text{see qn 3 for } \int \sec^3 \theta \, d\theta) \\
 &= 25 \sqrt{3} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + c \\
 &= \frac{25\sqrt{3}}{2} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) + c \\
 &= \frac{25\sqrt{3}}{2} \left(\frac{x}{5} \frac{\sqrt{x^2 - 25}}{5} - \ln \left| \frac{x + \sqrt{x^2 - 25}}{5} \right| \right) + c \\
 &= \frac{\sqrt{3}}{2} x \sqrt{x^2 - 25} - \frac{25\sqrt{3}}{2} \ln |x + \sqrt{x^2 - 25}| + k
 \end{aligned}$$

$$\begin{aligned}
 9 \int \frac{7}{\sqrt{7x^2 + 28}} \, dx &= \sqrt{7} \int \frac{1}{\sqrt{x^2 + 4}} \, dx \\
 x &= 2 \tan \theta \quad dx = 2 \sec^2 \theta \, d\theta \\
 x^2 + 4 &= 4 (\tan^2 \theta + 1) = 4 \sec^2 \theta \\
 \sqrt{x^2 + 4} &= 2 \sec \theta
 \end{aligned}$$

$$\begin{aligned}\int \frac{7}{\sqrt{7x^2+28}} dx &= \sqrt{7} \int \frac{1}{2\sec\theta} 2\sec^2\theta d\theta \\ &= \sqrt{7} \int \sec\theta d\theta \\ &= \sqrt{7} \ln |\sec\theta + \tan\theta| + c \\ &= \sqrt{7} \ln \left| \frac{\sqrt{x^2+4}+x}{2} \right| + c \\ &= \sqrt{7} \ln |\sqrt{x^2+4}+x| + k\end{aligned}$$

Exercise 9V



$P(x, \sin x)$

$$A = (\pi - 2x) \sin x$$

Max. area = 1.12 when $x = 0.71046$

$$\text{length} = \pi - 2x = 1.72$$

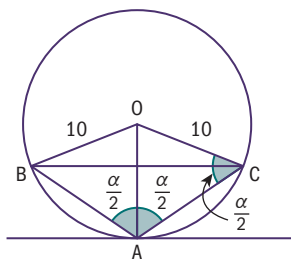
$$\text{height} = \sin x = 0.652$$

2 $A(1, -1)$ $P(x, \cos x)$

$$AP = \sqrt{(x-1)^2 + (\cos x + 1)^2}$$

Min. distance = 1.11 (at the point (1.78, -0.025))

3 a



In $\triangle AOC$, angle $AOC = \pi - \alpha$

$$\begin{aligned}\text{area } \triangle AOC &= \frac{1}{2} \cdot 10 \cdot 10 \sin(\pi - \alpha) \\ &= 50 \sin \alpha\end{aligned}$$

Similarly, area $AOB = 50 \sin \alpha$

In $\triangle BOC$, angle $BOC = 2\pi - 2\alpha$

$$\begin{aligned}\text{area } BOC &= \frac{1}{2} \cdot 10 \cdot 10 \sin(2\pi - 2\alpha) \\ &= -50 \sin 2\alpha\end{aligned}$$

$$\therefore \text{area } ABC = 50 \sin \alpha + 50 \sin \alpha - (-50 \sin 2\alpha)$$

$$A(\alpha) = 100 \sin \alpha + 50 \sin 2\alpha$$

$$= 100 \sin \alpha + 100 \sin \alpha \cos \alpha$$

$$A(\alpha) = 100(1 + \cos \alpha) \sin \alpha$$

b For maximum area, $\alpha = 1.05$

4 a $t = 2\sqrt{\frac{15}{g \sin 2\theta}}$ we require t to be a minimum

$$\theta = 0.785 = \frac{\pi}{4}$$

b $\theta = 0.7854$ $1 = \frac{15}{\cos \theta}$
 $= 21.213\text{m}$ (nearest mm)

5 a $d(t) = \sin\left(\frac{\pi t}{6}\right) + \cos\left(\frac{\pi t}{6}\right)$

$$v(t) = \frac{\pi}{6} \cos\left(\frac{\pi t}{6}\right) - \frac{\pi}{6} \sin\left(\frac{\pi t}{6}\right)$$

$$a(t) = \frac{-\pi^2}{36} \sin\left(\frac{\pi t}{6}\right) - \frac{\pi^2}{36} \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore a(t) = \frac{-\pi^2}{36} d(t)$$

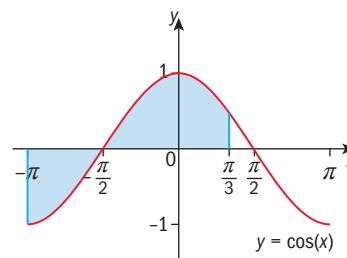
$\therefore$ acceleration is proportional to displacement.

b Max. speed = 0.740 ms^{-1} when $t = 4.50 \text{ s}$ (velocity in negative and a minimum at this time).

6 Min. height = 1.82 m when $x = 1.31 \text{ m}$
 $\therefore$ the first pole is nearer to the point of minimum height.

Exercise 9W

1 a



$$\int_{-\pi}^{-\frac{\pi}{2}} \cos x dx = [\sin x]_{-\pi}^{-\frac{\pi}{2}}$$

$$= \sin\left(-\frac{\pi}{2}\right) - \sin(-\pi)$$

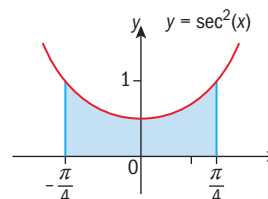
$$= -1 - 0 = -1$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{2}\right)$$

$$= \frac{\sqrt{3}}{2} + 1$$

$$\therefore \text{total area} = 2 + \frac{\sqrt{3}}{2}$$

b

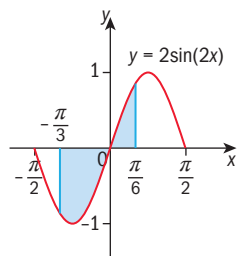


$$\text{Area} = 2 \int_0^{\frac{\pi}{4}} \sec^2 x dx$$

$$= 2[\tan x]_0^{\frac{\pi}{4}}$$

$$= 2 \tan \frac{\pi}{4} = 2$$

c



$$\int_{-\pi/2}^0 \sin 2x \, dx = [-\cos 2x]_{-\pi/2}^0$$

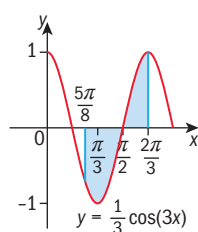
$$= -\cos 0 + \cos\left(-\frac{2\pi}{2}\right) = -1 + \left(-\frac{1}{2}\right) = -\frac{3}{2}$$

$$\int_0^{\pi/2} 2 \sin 2x \, dx = [-\cos 2x]_0^{\pi/2}$$

$$= -\cos \pi + \cos 0 = -(-1) + 1 = \frac{1}{2}$$

area = 2

d



$$\int_{5\pi/18}^{\pi/2} \frac{1}{3} \cos 3x \, dx = \frac{1}{9} [\sin 3x]_{5\pi/18}^{\pi/2}$$

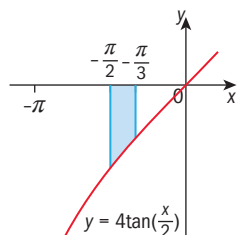
$$= \frac{1}{9} \left(\sin \frac{3\pi}{2} - \sin \frac{5\pi}{6} \right) = \frac{1}{9} \left(-1 - \frac{1}{2} \right) = -\frac{1}{6}$$

$$\int_{\pi/2}^{2\pi/3} \frac{1}{3} \cos 3x \, dx = \frac{1}{9} [\sin 3x]_{\pi/2}^{2\pi/3} = \frac{1}{9} (\sin 2\pi - \sin \frac{3\pi}{2})$$

$$= \frac{1}{9} (0 - (-1)) = \frac{1}{9}$$

$$\text{area} = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$$

e



$$\int_{-\pi/2}^{-\pi/3} 4 \tan\left(\frac{x}{2}\right) \, dx = 8 \left[\ln \left| \sec \frac{x}{2} \right| \right]_{-\pi/2}^{-\pi/3}$$

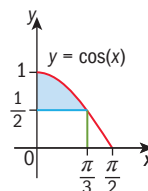
$$= 8 \left(\ln \left| \sec \left(-\frac{\pi}{6}\right) \right| - \ln \left| \sec \left(-\frac{\pi}{4}\right) \right| \right)$$

$$= 8 \left(\ln \frac{2}{\sqrt{3}} - \ln(\sqrt{2}) \right)$$

$$= 8 \ln \sqrt{\frac{2}{3}} = -8 \ln \sqrt{\frac{3}{2}} = -4 \ln \left(\frac{3}{2} \right)$$

$$\text{area} = 4 \ln \left(\frac{3}{2} \right)$$

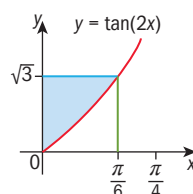
2 a



$$\text{Area} = \int_0^{\pi/3} \cos x \, dx = \left[\sin x \right]_0^{\pi/3}$$

$$= \sin \frac{\pi}{3} - \sin 0 = \frac{\sqrt{3}}{2}$$

b

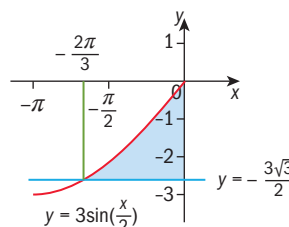


$$\text{Area} = \int_0^{\pi/4} \tan 2x \, dx = \left[-\frac{1}{2} \ln |\sec 2x| \right]_0^{\pi/4}$$

$$= -\frac{1}{2} \left(\ln |\sec \frac{\pi}{2}| - \ln |\sec 0| \right)$$

$$= -\frac{1}{2} (\ln 2 - \ln 1) = -\frac{1}{2} \ln 2$$

c



$$3 \sin \frac{x}{2} = -3 \frac{\sqrt{3}}{2}$$

$$\sin \frac{x}{2} = -\frac{\sqrt{3}}{2}$$

$$\frac{x}{2} = -\frac{\pi}{3}$$

$$x = -\frac{2\pi}{3}$$

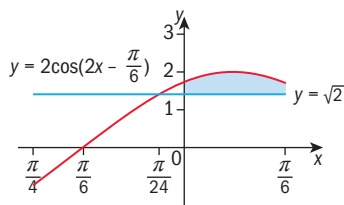
$$\int_{-2\pi/3}^{-\pi/2} 3 \sin \frac{x}{2} \, dx = -6 \left[\cos \left(\frac{x}{2} \right) \right]_{-2\pi/3}^{-\pi/2}$$

$$= -6 \left(\cos 0 - \cos \left(-\frac{\pi}{3} \right) \right)$$

$$= -6 \left(1 - \frac{1}{2} \right) = -3$$

$$\text{Area} = \frac{2\pi}{3} \left(\frac{3\sqrt{3}}{2} \right) - 3 = \sqrt{3}\pi - 3$$

d



$$2\cos\left(2x - \frac{\pi}{6}\right) = \sqrt{2}$$

$$2x - \frac{\pi}{6} = \frac{\pi}{4}$$

$$2x = \frac{5\pi}{12}$$

$$x = \frac{5\pi}{24}$$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{5\pi}{24}} 2\cos\left(2x - \frac{\pi}{6}\right) dx - \frac{5\pi}{24}(\sqrt{2}) \\ &= \left[\sin\left(2x - \frac{\pi}{6}\right)\right]_0^{\frac{5\pi}{24}} - \frac{5\pi}{24}(\sqrt{2}) \\ &= \sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{6}\right) - \frac{5\pi\sqrt{2}}{24} \\ &= \frac{\sqrt{2}+1}{2} - \frac{5\pi\sqrt{2}}{24} \end{aligned}$$

e $\tan \frac{x}{3} = \frac{1}{\sqrt{3}}$

$$\frac{x}{3} = \frac{\pi}{6}$$

$$x = \frac{\pi}{2}$$

$$\begin{aligned} \text{Area} &= \frac{\pi}{2\sqrt{3}} - \int_0^{\frac{\pi}{2}} \tan \frac{x}{3} dx \\ &= \frac{\pi}{2\sqrt{3}} - \left[3\ln\left|\sec \frac{x}{6}\right|\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2\sqrt{3}} - 3\left(\ln\left|\sec \frac{\pi}{6}\right| - \ln|\sec 0|\right) \\ &= \frac{\pi}{2\sqrt{3}} - 3\ln \frac{2}{\sqrt{3}} \end{aligned}$$

3 $\text{Area} = \int_0^{2.51327} (\cos \frac{x}{2} - \cos 2x) dx = 2.38$

4 $\frac{dy}{dx} = \sec^2 x$

$$\left(\frac{\pi}{4}, 1\right) \frac{dy}{dx} = 2$$

Tangent: $y - 1 = 2\left(x - \frac{\pi}{4}\right)$

$$y = 2x - \frac{\pi}{2} + 1$$

if $y = 0$, $2x = \frac{\pi}{2} - 1$

$$x = \frac{\pi}{4} - \frac{1}{2}$$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{4}} \tan x dx - \frac{1}{2}\left(\frac{1}{2}\right) \\ &= \left[\ln|\sec x|\right]_0^{\frac{\pi}{4}} - \frac{1}{4} = \ln\left|\sec \frac{\pi}{4}\right| - \ln|\sec 0| - \frac{1}{4} \\ &= \ln\sqrt{2} - \frac{1}{4} = \frac{1}{2}\ln 2 - \frac{1}{4} \end{aligned}$$

5 $\text{Area} = \int_0^{2.57915} |2\sin x - e^{\frac{x}{2}-4} - 1| dx = 1.55$

6 $y = \frac{8}{4+x^2} \quad y = \frac{x^2}{4}$

a $\frac{8}{4+x^2} = \frac{x^2}{4} \Rightarrow 32 = 4x^2 + x^4$

$$x^4 + 4x^2 - 32 = 0$$

$$(x^2 + 8)(x^2 - 4) = 0$$

$$x^2 = 4, x = \pm 2 \quad (2, 1), (-2, 1)$$

b, c $\text{Area} = \int_{-2}^2 \left(\frac{8}{4+x^2} - \frac{x^2}{4}\right) dx$

$$\begin{aligned} &= \left[\frac{8}{2}\arctan\left(\frac{x}{2}\right) - \frac{x^3}{12}\right]_{-2}^2 \\ &= \left(4\arctan 1 - \frac{2}{3}\right) - \left(4\arctan(-1) + \frac{2}{3}\right) \\ &= \left(4\frac{\pi}{4} - \frac{2}{3}\right) - \left(-4\frac{\pi}{4} + \frac{2}{3}\right) \\ &= 2\pi - \frac{4}{3} \end{aligned}$$

Exercise 9X

1 a $v = \pi \int_0^{\frac{\pi}{2}} \cos x dx = \pi[\sin x]_0^{\frac{\pi}{2}} = \pi\left(\sin \frac{\pi}{2} - \sin 0\right)$

$$v = \pi$$

b $v = \pi \int_0^{\frac{\pi}{2}} \sec^2 x dx = \pi[\tan x]_0^{\frac{\pi}{4}} = \pi\left(\tan \frac{\pi}{4} - \tan 0\right)$

$$v = \pi$$

c $v = \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2 x 2dx = \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \cos 2x) dx$

$$\begin{aligned} &= \frac{\pi}{2} \left[x + \frac{1}{2}\sin 2x \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\ &= \frac{\pi}{2} \left[\left(\frac{5\pi}{6} + \frac{1}{2}\sin \frac{5\pi}{3}\right) - \left(\frac{\pi}{6} + \frac{1}{2}\sin \frac{\pi}{3}\right) \right] \\ &= \frac{\pi}{2} \left[\frac{5\pi}{6} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right] = \frac{\pi}{2} \left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right] \\ &= \pi \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) \end{aligned}$$

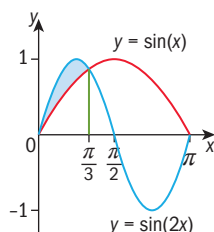
d $v = \pi \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^2 x dx = \frac{\pi}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (1 - \cos 2x) dx$

$$\begin{aligned} &= \frac{\pi}{2} \left[x - \frac{1}{2}\sin 2x \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\ &= \frac{\pi}{2} \left[\left(\frac{2\pi}{3} - \frac{1}{2}\sin \frac{4\pi}{3}\right) - \left(\frac{\pi}{3} - \frac{1}{2}\sin \frac{2\pi}{3}\right) \right] \\ &= \frac{\pi}{2} \left[\frac{2\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right] \\ &= \frac{\pi}{2} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad v &= \pi \int_0^1 \sin^2 y \, dy = \frac{\pi}{2} \int_0^1 (1 - \cos 2y) \, dy \\
 &= \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^1 \\
 &= \frac{\pi}{2} \left[\left(1 - \frac{1}{2} \sin 2 \right) - 0 \right] \\
 &= \frac{\pi}{2} \left(1 - \frac{1}{2} \sin 2 \right)
 \end{aligned}$$

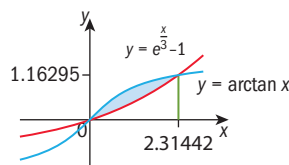
$$\begin{aligned}
 b \quad y &= \arcsin x, \quad 0 \leq x \leq 1 \\
 \Rightarrow x &= \sin y, \quad 0 \leq y \leq \frac{\pi}{2} \\
 v &= \pi \int_0^{\frac{\pi}{2}} \sin^2 y \, dy = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2y) \, dy \\
 &= \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{2} \left[\frac{\pi}{2} - \frac{1}{2} \sin \pi - (0) \right] \\
 &= \frac{\pi^2}{4}
 \end{aligned}$$

3 b



$$\begin{aligned}
 v &= \pi \int_0^{\frac{\pi}{3}} (\sin^2 2x - \sin^2 x) \, dx \\
 &= \pi \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2} \cos 4x - \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \frac{\pi}{2} \int_0^{\frac{\pi}{3}} (\cos 2x - \cos 4x) \, dx \\
 &= \frac{\pi}{2} \left[\frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{3}} \\
 &= \frac{\pi}{2} \left[\frac{1}{2} \sin \frac{2\pi}{3} - \frac{1}{4} \sin \frac{4\pi}{3} \right] \\
 &= \frac{\pi}{2} \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} \right) = \frac{3\pi\sqrt{3}}{16}
 \end{aligned}$$

c



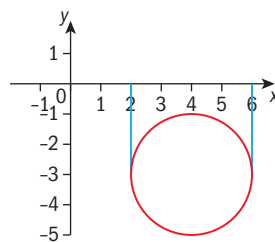
$$v = \pi \int_0^{2.31442} \left((\arctan x)^2 - (e^{\frac{x}{3}} - 1)^2 \right) dx = 2.35$$

 d $y = \arctan x \Rightarrow x^2 + 1 = \tan^2 y$

$$\begin{aligned}
 y &= e^{\frac{x}{3}} - 1 \Rightarrow e^{\frac{x}{3}} = y + 1 \Rightarrow x = 3 \ln(y + 1) \\
 v &= \pi \int_0^{1.16295} ((3 \ln(y + 1))^2 - \tan^2 y) \, dy = 4.18
 \end{aligned}$$

Exercise 9Y

1



$$(x - 4)^2 + (y + 3)^2 = 4$$

$$\begin{aligned}
 (y + 3)^2 &= 4 - (x - 4)^2 \\
 y + 3 &= \pm \sqrt{4 - (x - 4)^2} \\
 y &= -3 \pm \sqrt{4 - (x - 4)^2}
 \end{aligned}$$

$$\begin{aligned}
 V &= \pi \int_2^6 \left((-3 - \sqrt{4 - (x - 4)^2})^2 - (-3 + \sqrt{4 - (x - 4)^2})^2 \right) dx \\
 &= \pi \int_2^6 \left(12\sqrt{4 - (x - 4)^2} \right) dx \\
 &= 12\pi \int_2^6 \sqrt{4 - (x - 4)^2} \, dx
 \end{aligned}$$

$$\text{Let } x - 4 = 2\sin\theta \Rightarrow dx = 2\cos\theta \, d\theta$$

$$x = 2 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$$

$$x = 6 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\sqrt{4 - (x - 4)^2} = 2\cos\theta$$

$$\begin{aligned}
 v &= 12\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta \cdot 2\cos\theta \, d\theta \\
 &= 24\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta \\
 &= 24\pi \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
 \end{aligned}$$

$$= 24\pi \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$v = 24\pi^2$$

 2 $(x - 4)^2 + (y + 3)^2 = 4$

$$(x - 4)^2 = 4 - (y + 3)^2$$

$$x - 4 = \pm \sqrt{4 - (y + 3)^2}$$

$$x = 4 \pm \sqrt{4 - (y + 3)^2}$$

$$\begin{aligned}
 v &= \pi \int_{-5}^{-1} \left((4 + \sqrt{4 - (y + 3)^2})^2 - (4 - \sqrt{4 - (y + 3)^2})^2 \right) dy \\
 &= \pi \int_{-5}^{-1} 16\sqrt{4 - (y + 3)^2} \, dy
 \end{aligned}$$

$$\text{Let } y + 3 = 2\sin\theta \Rightarrow dy = 2\cos\theta \, d\theta$$

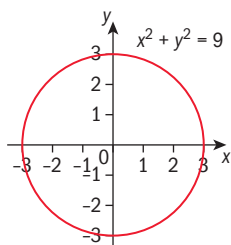
$$y = -5 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$$

$$y = -1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\sqrt{4 - (y + 3)^2} = 2\cos\theta$$

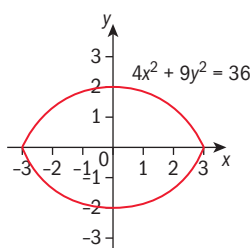
$$v = 16\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta \cdot 2\cos\theta \, d\theta = 32\pi^2 \text{ (see qn. 1)}$$

3



$$\begin{aligned}
 y^2 &= 9 - x^2 \\
 v &= \pi \int_{-3}^3 (9 - x^2) dx \\
 &= \pi \left[9x - \frac{x^3}{3} \right]_{-3}^3 \\
 v &= \pi \left[\left(27 - \frac{27}{3} \right) - \left(-27 + \frac{27}{3} \right) \right] \\
 v &= 36\pi
 \end{aligned}$$

4



$$\begin{aligned}
 4x^2 + 9y &= 36 \\
 y^2 &= 4 - \frac{4}{9}x^2 \\
 v &= \pi \int_{-3}^3 \left(4 - \frac{4}{9}x^2 \right) dx \\
 v &= \pi \left[4x - \frac{4x^3}{27} \right]_{-3}^3 \\
 &= \pi [(12 - 4) - (-12 + 4)] = 16\pi
 \end{aligned}$$

5 $x^2 = 9 - \frac{9}{4}y^2$, $v = \pi \int_{-2}^2 \left(9 - \frac{9}{4}y^2 \right) dy$

$$\begin{aligned}
 v &= \pi \left[9y - \frac{3}{4}y^3 \right]_{-2}^2 \\
 \pi [(18 - 6) - (-18 + 6)] &= 24\pi
 \end{aligned}$$



Review exercise

1 a $f(x) = (2x + 3)\sin x$

$$\Rightarrow f'(x) = 2\sin x + (2x + 3)\cos x$$

b $g(x) = e^x \cos 3x$

$$\Rightarrow g'(x) = e^x \cos 3x + e^x \cdot (-\sin 3x) \cdot 3$$

$$= e^x (\cos 3x - 3\sin 3x)$$

c $h(x) = \frac{\tan x}{2x^2} = \frac{1}{2} \tan x \cdot x^{-2}$

$$\begin{aligned}
 \Rightarrow h'(x) &= \frac{1}{2} \sec^2 x \cdot x^{-2} + \frac{1}{2} \tan x \cdot (-2x^{-3}) \\
 &= \frac{x - 2\sin x \cos x}{2x^3 \cos^2 x} \\
 &= \frac{x - \sin 2x}{2x^3 \cos^2 x}
 \end{aligned}$$

2 $\sin y + e^{2x} = 1 \Rightarrow \cos y \cdot y' + e^{2x} \cdot 2 = 0$

$$\Rightarrow y' = \frac{-2e^{2x}}{\cos y}$$

$$\Rightarrow m = y'(0) = \frac{-2e^0}{\cos 0} = -2$$

$$T: y - 0 = -2(x - 0) \Rightarrow y = -2x$$

3 $\int_{\frac{\pi}{4}}^m \sec^2 x dx = [\tan x]_{\frac{\pi}{4}}^m = \tan m - \tan \frac{\pi}{4}$

$$= \tan m - 1 = 2 \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right)$$

$$\Rightarrow \tan m - 1 = \sqrt{3} - 1 \Rightarrow \tan m = \sqrt{3} \Rightarrow m = \frac{\pi}{3}$$

4 a $\int (2x - 5)e^{2x} dx =$ Let $2x - 5 = u \Rightarrow 2dx = du$

$$e^{2x} dx = dv \Rightarrow \frac{1}{2} e^{2x} = v,$$

$$\begin{aligned}
 \frac{2x-5}{2} e^{2x} - \int e^{2x} dx &= \left(x - \frac{5}{2} \right) e^{2x} - \frac{1}{2} e^{2x} + c \\
 &= (x - 3) e^{2x} + c
 \end{aligned}$$

b

	$dv = \cos x dx$	sign
$u = x^2 - 5$	$v = \sin x$	+
$2x$	$-\cos x$	-
2	$-\sin x$	+

$$\begin{aligned}
 (x^2 - 5x) \cos x dx &= (x^2 - 5x) \sin x \\
 &\quad + 2x \cos x - 2 \sin x + c; \\
 &= (x^2 - 5x - 2) \sin x + 2x \cos x + c
 \end{aligned}$$

c

	$dv = e^x dx$	sign
$u = \cos x$	$v = e^x$	+
$-3\sin 3x$	e^x	-
$-9\cos 3x$	e^x	+

$$\begin{aligned}
 \int e^x \cos 3x dx &= e^x \cos 3x + e^x 3 \sin 3x \\
 &\quad - 9 \int e^x \cos 3x dx. \\
 10 \int e^x \cos 3x dx &= e^x \cos 3x + 3e^x \sin 3x \\
 \Rightarrow \int e^x \cos 3x dx &= \frac{e^x}{10} (\cos 3x + 3 \sin 3x)
 \end{aligned}$$

5 $A = \frac{1}{2} d^2 \Rightarrow \frac{dA}{dt} = d \cdot \frac{dd}{dt}$

$$\Rightarrow \frac{dA}{dt} = \sqrt{5} \cdot 0.2 = \frac{\sqrt{5}}{5} \text{ cm}^2/\text{s}.$$

6 The curve $y = e^{2x-1}$ is given.

a $y = e^{2x-1} \Rightarrow y' = e^{2x-1} \cdot 2$

$$m = y'(x_0) = e^{2x_0-1} \cdot 2 \Rightarrow T: y - y_0 = m(x - x_0)$$

$$y = 2e^{2x_0-1}(x - x_0) + y_0 \Rightarrow$$

$$y = 2e^{2x_0-1}x - 2e^{2x_0-1}x_0 + e^{2x_0-1}$$

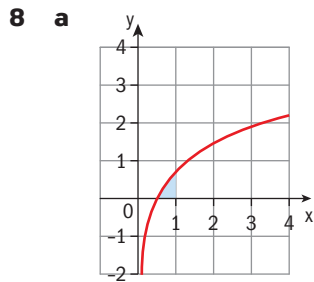
$$e^{2x_0-1}(-2x_0 + 1) = 0 \Rightarrow x_0 = \frac{1}{2}$$

$$T: y = 2x$$

$$\begin{aligned} \text{b } \int_0^{\frac{1}{2}} (e^{2x-1} - 2x) dx &= \left[\frac{1}{2} e^{2x-1} - x^2 \right]_0^{\frac{1}{2}} \\ &= \frac{1}{2} - \frac{1}{4} - \frac{1}{2e} = \frac{e-2}{4e} \end{aligned}$$

$$\begin{aligned} \text{c } \int_0^{\frac{1}{2}} ((e^{2x-1})^2 - (2x)^2) dx &= \pi \int_0^{\frac{1}{2}} (e^{4x-2} - 4x^2) dx \\ &= \pi \left[\frac{1}{4} e^{4x-2} - \frac{4}{3} x^3 \right]_0^{\frac{1}{2}} \\ &= \pi \left(\frac{1}{4} - \frac{1}{6} - \frac{1}{4e^2} \right) = \frac{(e^2-3)\pi}{12e^2} \end{aligned}$$

$$\begin{aligned} \text{7 Let } x &= 3\cos\theta \Rightarrow dx = -3\sin\theta d\theta, \quad \theta = \arccos\left(\frac{x}{3}\right) \\ \sqrt{9-x^2} &= \sqrt{9-9\cos^2\theta} = 3\sin\theta \\ \int \sqrt{9-x^2} dx &= -9 \int \sin^2\theta d\theta - 9 \int \frac{1-\cos 2\theta}{2} d\theta \\ &= -9 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] \\ &= \frac{9}{2} \sin\left(\arccos\left(\frac{x}{3}\right)\right) \cos\left(\arccos\left(\frac{x}{3}\right)\right) - \frac{9}{2} \arccos\left(\frac{x}{3}\right) + c \\ &= \frac{x}{2} \sqrt{9-x^2} - \frac{9}{2} \arccos\left(\frac{x}{3}\right) + c \end{aligned}$$



$$\text{b } y = \ln(2x) \Rightarrow x = \frac{1}{2} e^y, \quad x = 1 \Rightarrow y = \ln 2$$

$$\begin{aligned} \pi \int_0^{\ln 2} \left(1 - \left(\frac{1}{2} e^y \right)^2 \right) dy &= \pi \int_0^{\ln 2} \left(1 - \frac{e^{2y}}{4} \right) dy \\ &= \pi \left[y - \frac{1}{8} e^{2y} \right]_0^{\ln 2} \\ &= \pi \left(\ln 2 - \frac{4-1}{8} \right) = \frac{(8\ln 2-3)\pi}{8} \end{aligned}$$

$$\begin{aligned} \text{9 a } s &= \int_0^k v dt = \int_0^k 5e^{-\frac{2t}{3}} dt = 5 \left[-\frac{3}{2} e^{-\frac{2t}{3}} \right]_0^k \\ &= \frac{15}{2} \left(1 - e^{-\frac{2k}{3}} \right) \end{aligned}$$

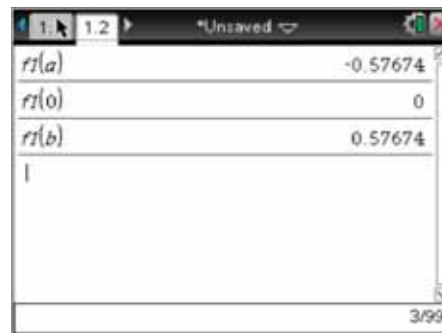
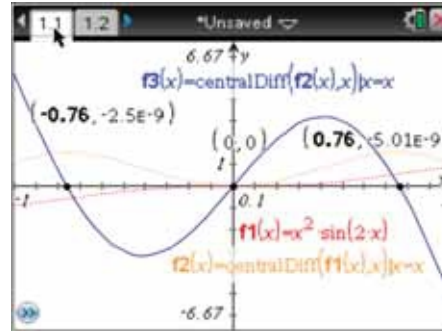
$$\text{b } \lim_{k \rightarrow \infty} s = \lim_{k \rightarrow \infty} \left(\frac{15}{2} - \frac{15}{2} e^{-\frac{2k}{3}} \right) = \frac{15}{2} - 0 = 7.5 \text{ m}$$

$$\begin{aligned} \text{10 } x^2 y^3 &= \cos(\pi x) \Rightarrow 2xy^3 + x^2 3y^2 y' = -\sin(\pi x) \cdot \pi \\ 2 \cdot 1 \cdot (-1)^3 + 1^2 \cdot 3 \cdot (-1)^2 \cdot m_T &= -\sin(\pi) \cdot \pi \\ \Rightarrow m_T &= \frac{2}{3} \Rightarrow m_N = -\frac{3}{2} \\ N: y+1 &= -\frac{3}{2}(x-1) \Rightarrow y = -\frac{3}{2}x + \frac{1}{2} \end{aligned}$$



Review exercise

- 1 We need to find the zeros of the second derivative of the function $y = x^2 \sin 2x$, $-1 \leq x \leq 1$. We store the variables a and b and then to find the y -coordinates of the points of inflexion we input those values of x in the original function.



So the points of inflexion are $(-0.760, -0.577)$, $(0, 0)$ and $(0.760, 0.577)$.

$$\text{2 } y^3 = \cos x \Rightarrow 3y^2 y' = -\sin x \Rightarrow y' = -\frac{\sin x}{3y^2}$$

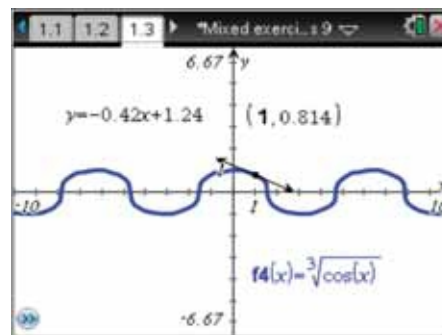
$$\text{When } x = 1 \Rightarrow y^3 = \cos 1 \Rightarrow y = \sqrt[3]{\cos 1} = 0.814$$

$$m = -\frac{\sin 1}{3\sqrt[3]{\cos^2 1}} = -0.423$$

$$\begin{aligned} T: y - 0.814 &= -0.423(x - 1) \Rightarrow \\ y &= -0.423x + 1.24 \end{aligned}$$

Checking:

First we find the explicit form of the curve and graph it on a GDC. $y^3 = \cos x \Rightarrow y = \sqrt[3]{\cos x}$



- 3 Find the value of a , $0 < a < 1$, such that

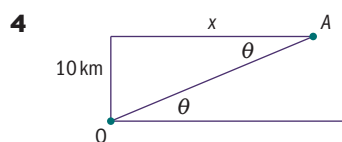
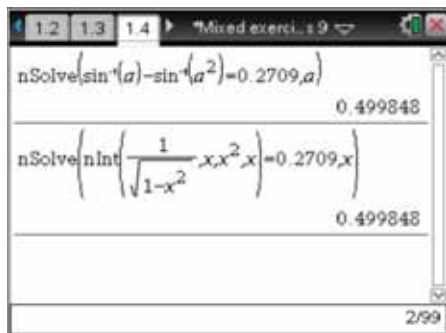
$$\int_a^1 \frac{1}{\sqrt{1-x^2}} dx = 0.2709$$

$$\int_a^1 \frac{1}{\sqrt{1-x^2}} dx = [\arcsin x]_a^1 = \arcsin 1 - \arcsin a$$

$$= 0.2709$$

We use a GDC to solve this equation, $a = 0.500$.

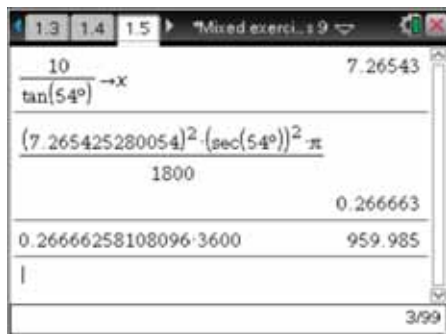
The result can be obtained directly on a calculator by using numerical integration.



$$\tan \theta = \frac{10}{x} \Rightarrow x = \frac{10}{\tan \theta} = \frac{10}{\tan 54^\circ} = 7.27$$

$$\tan \theta = \frac{10}{x} \Rightarrow \sec^2 \theta \cdot \frac{d\theta}{dt} = -\frac{10}{x^2} \cdot \frac{dx}{dt}$$

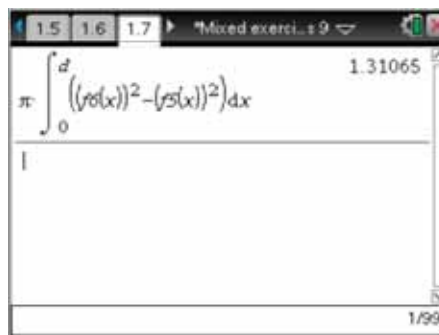
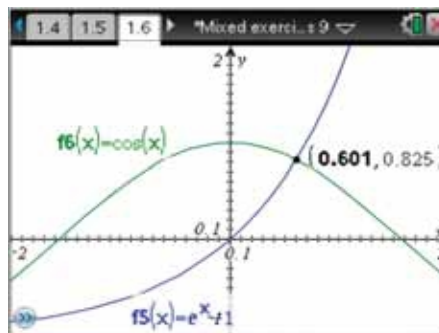
$$\frac{dx}{dt} = -\frac{x^2 \sec^2 \theta}{10} \cdot \frac{d\theta}{dt} = -\frac{7.27^2 \cdot \sec^2 54^\circ}{10} \cdot \frac{\pi}{180} = -0.267 \text{ km}^{-1}$$



So the speed of the plane is 960 km/h.

- 5 First we graph the functions and identify the region. Then we find the point of intersection between the curves and store the x -coordinate to a variable called d .

$$V = \pi \int_0^{0.601} (\cos^2 x - (e^x - 1)^2) dx = 1.31$$



10

Modeling randomness

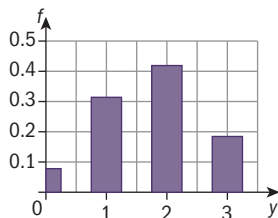
Skills check

- 1 a $\frac{1}{8}$ b $\frac{5}{36}$
 2 a 0.8 b $0.2 + 0.3 - 0.1 = 0.4$

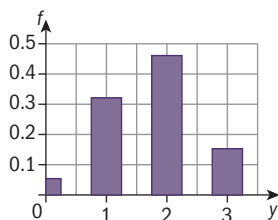
Exercise 10A

- 1 a No, because $\sum f(x) = 1.1$, not 1
 b No, as above
 c No, because $f(-1)$ is negative.
 2 a $a = 1 - 0.88 = 0.12$
 b $P(1 \leq X \leq 3) = P(1) + P(2) + P(3) = 0.87$
 c $P(X \leq 3) = 0.87$
 3 a $\sum_0^3 f(t) = 1 \Rightarrow 4k + 3k + 2k + k = 1 \Rightarrow k = \frac{1}{10}$
 b $P(1 \leq T < 3) = P(1) + P(2) = 3k + 2k = \frac{1}{2}$

x	0	1	2	3
P(X = x)	$\frac{27}{343}$	$\frac{108}{343}$	$\frac{144}{343}$	$\frac{64}{343}$



x	0	1	2	3
P(X = x)	$\frac{120}{2184}$	$\frac{702}{2184}$	$\frac{1008}{2184}$	$\frac{336}{2184}$



Exercise 10B

- 1 a $E(R) = 1 \times \frac{1}{5} + 5 \times \frac{2}{5} + 10 \times \frac{2}{5} = \frac{31}{5} = 6.2$
 b $E(R^2) = 1 \times \frac{1}{5} + 25 \times \frac{2}{5} + 100 \times \frac{2}{5} = 50.2$
 c $\text{Var}(R) = E(R^2) - [E(R)]^2 = 50.2 - 6.2^2 = 11.76$
 d Standard deviation $= \sqrt{11.76} = 3.43(3\text{sf})$

2 a $P(X \geq 2) = 3P(X < 2)$
 $\Rightarrow a + 3b = 3 \times 2a \Rightarrow 3b = 5a$
 But $\sum P(X) = 1$ so $3a + 3b = 1$
 $\therefore 1 - 3a = 5a$ so $a = \frac{1}{8}$ and $b = \frac{5}{24}$

b $E(X) = 0 \times \frac{1}{8} + 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{5}{24} + 4$
 $\times \frac{5}{24} + 5 \times \frac{5}{24}$
 $= \frac{23}{8}$

$E(X^2) = 0 \times \frac{1}{8} + 1 \times \frac{1}{8} + 4 \times \frac{1}{8} + 9 \times \frac{5}{24} + 16$
 $\times \frac{5}{24} + 25 \times \frac{5}{24}$
 $= \frac{265}{24}$

c $\text{Var}(X) = E(X^2) - E(X)^2$
 $= \frac{265}{24} - \left(\frac{23}{8}\right)^2$
 $= \frac{533}{192}$

3 a $P(\text{bottom} > \text{top}) = P(\text{top} > \text{bottom})$.
 Since bottom cannot equal top,
 $P(\text{bottom} > \text{top}) = \frac{1}{2}$

b $P(S = 4) = P(1, 3) + P(3, 1)$
 $= \frac{1}{10} \times \frac{1}{9} + \frac{1}{10} \times \frac{1}{9} = \frac{1}{45}$

x	P(S = x)	x	P(S = x)
3	$\frac{1}{45}$	11	$\frac{1}{9}$
4	$\frac{1}{45}$	12	$\frac{4}{45}$
5	$\frac{2}{45}$	13	$\frac{4}{45}$
6	$\frac{2}{45}$	14	$\frac{3}{45}$
7	$\frac{3}{45}$	15	$\frac{3}{45}$
8	$\frac{3}{45}$	16	$\frac{2}{45}$
9	$\frac{4}{45}$	17	$\frac{2}{45}$
		18	$\frac{1}{45}$
10	$\frac{4}{45}$	19	$\frac{1}{45}$

$$\begin{aligned} \text{d } E(S) &= 3 \times \frac{1}{45} + 4 \times \frac{1}{45} + \dots + 18 \times \frac{1}{45} + 19 \times \frac{1}{45} \\ &= \frac{495}{45} = 11 \quad (\text{also obvious from symmetry}) \end{aligned}$$

$$\begin{aligned} \text{Var}(S) &= \frac{1}{45} (9 + 16 + 25 \times 2 + 36 \times 2 + 49 \\ &\quad \times 3 + 64 \times 3 + 81 \times 4 + 100 \times 4 + 121 \\ &\quad \times 5 + 144 \times 4 + 169 \times 4 + 196 \times 3 + 225 \\ &\quad \times 3 + 256 \times 2 + 289 \times 2 + 324 + 361) - 11^2 \\ &= \frac{6105}{45} = 14\frac{2}{3} \end{aligned}$$

4 Let £ A be the amount paid.

Then if W = your winnings,

$$E(W) = 20 \times 0.2 + 10 \times 0.4 + (-A) \times 0.4$$

$$E(W) = 0, \quad \text{so } 4 + 4 - 0.4A = 0$$

$$\Rightarrow 0.4A = 8 \Rightarrow A = 20$$

So you pay £20

$$5 \text{ a } \sum_{c=1}^7 f(t) = 1 \Rightarrow k + 4k + 9k + 16k + 9k + 4k + k = 1$$

$$\Rightarrow 44k = 1 \Rightarrow k = \frac{1}{44}$$

$$\text{b } P(T = 4) = k(8 - 4)^2 = 16k = \frac{16}{44} = \frac{4}{11}$$

$$P(T \leq 4) = k + 4k + 9k + 16k = \frac{30}{44} = \frac{15}{22}$$

$$\begin{aligned} P(T = 4 | T \leq 4) &= \frac{P(T = 4 \text{ and } T \leq 4)}{P(T \leq 4)} = \frac{P(T = 4)}{P(T \leq 4)} \\ &= \frac{\frac{4}{11}}{\frac{15}{22}} = \frac{8}{15} \end{aligned}$$

$$\text{c } E(T) = \sum_1^7 tf(t)$$

$$\begin{aligned} &= 1 \times k + 2 \times 4k + 3 \times 9k + 4 \times 16k + 5 \\ &\quad \times 9k + 6 \times 4k + 7 \times k \\ &= \frac{176}{44} = 4 \end{aligned}$$

$$\text{Var}(T) = E(T^2) - 4^2$$

$$\begin{aligned} &= 1 \times k + 4 \times 4k + 9 \times 9k + 16 \times 16k + 25 \times 9k \\ &\quad + 36 \times 4k + 49 \times k - 16 \\ &= \frac{772}{44} - 16 = \frac{17}{11} \end{aligned}$$

d Mode of $T = 4$ (highest probability)

Exercise 10C

$$1 \text{ a } P(X \leq 15) = \frac{16}{15} = \frac{2}{5}$$

x	5	10	15	20	25	30
$f(x) = P(X \leq x)$	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{6}{15}$	$\frac{10}{15}$	$\frac{13}{15}$	1

c From table, median = 20

$$\begin{aligned} 2 \text{ a } P(\text{at least 3 lines}) &= P(L \geq 3) \\ &= 0.31 + 0.12 + 0.04 = 0.47 \end{aligned}$$

$$\begin{aligned} \text{b } E(L) &= 0 \times 0.07 + 1 \times 0.21 + 2 \times 0.25 + 3 \times 0.31 \\ &\quad + 4 \times 0.12 + 5 \times 0.04 \\ &= 2.32 \end{aligned}$$

$$\begin{aligned} \text{Var}(L) &= E(L^2) - 2.32^2 \\ &= 6.92 - 5.3824 = 1.5376 \end{aligned}$$

x	0	1	2	3	4	5
$f(x) = P(X \leq x)$	0.07	0.28	0.53	0.84	0.96	1

d From table, median = 2

x	2	3	4	5	6
$f(x) = P(X = x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
$f(x) = P(X \leq x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

$$\begin{aligned} \text{b } \text{Mean} = E(S) &= 2 \times \frac{1}{36} + 3 \times \frac{4}{36} + 4 \times \frac{10}{36} + 5 \times \frac{12}{36} + 6 \times \frac{9}{36} \\ &= \frac{2 + 12 + 40 + 60 + 54}{36} \\ &= \frac{168}{36} = 4\frac{2}{3} \end{aligned}$$

From table, median = 5, mode = 5

$$\begin{aligned} \text{c } \text{Var}(X) &= E(X^2) - \left(4\frac{2}{3}\right)^2 \\ &= 4 \times \frac{1}{36} + 9 \times \frac{4}{36} + 16 \times \frac{10}{36} + 25 \times \frac{12}{36} + 36 \times \frac{9}{36} \\ &= 22\frac{8}{9} - 21\frac{7}{9} = 1\frac{1}{9} \end{aligned}$$

$$\text{Standard deviation} = \sqrt{1\frac{1}{9}} = 1.05 \quad (3 \text{ sf})$$

$$\begin{aligned} 4 \text{ a } \sum_{k=1}^n kx &= 1 \Rightarrow k(1 + 2 + \dots + n) = 1 \\ &\Rightarrow k\frac{n}{2}(n+1) = 1 \Rightarrow k = \frac{2}{n^2 + n} \end{aligned}$$

$$\begin{aligned} \text{b } E(X) &= 1 \times k + 2 \times 2k + 3 \times 3k + \dots + n \times nk \\ &= k(1^2 + 2^2 + 3^2 + \dots + n^2) \\ &= k\frac{n}{6}(n+1)(2n+1) \\ &= \frac{2}{n(n+1)} \times \frac{n}{6}(n+1)(2n+1) \\ &= \frac{2n+1}{3} \end{aligned}$$

$$\begin{aligned} 5 \text{ a } \sum_1^\infty f(x) &= 1 \Rightarrow 3^{a-1} + 3^{a-2} + 3^{a-3} + \dots = 1 \\ &\Rightarrow 3^{a-1} = \left(1 + \frac{1}{3} + \frac{1}{9} + \dots\right) = 1 \\ &\Rightarrow 3^{a-1} \times \frac{1}{1 - \frac{1}{3}} = 1 \Rightarrow 3^{a-1} \times \frac{3}{2} = 1 \\ &\Rightarrow 3^a = 2 \Rightarrow a = \log_3 2 \end{aligned}$$

$$\begin{aligned}
 \text{b } F(x) &= P(X \leq x) = \sum_{r=1}^x 3^{a-r} \\
 &= 3^{a-1} + 3^{a-2} + \dots + 3^{a-x} \\
 &= 3^{a-1} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{x-1}} \right) \\
 &= 3^{a-1} \times \frac{1 - \left(\frac{1}{3}\right)^x}{1 - \frac{1}{3}} \\
 &= 3^{a-1} \times \frac{3}{2} \left(1 - \frac{1}{3^x} \right) = 1 - \frac{1}{3^x} \\
 \therefore F(x) &= 1 - 3^{-x}, x = 1, 2, 3
 \end{aligned}$$

Exercise 10D

1 $X \sim B\left(10, \frac{1}{2}\right)$

a $10C4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6 = \frac{210}{1024} = \frac{105}{512} \approx 0.205$

b $P(X \geq 6) = 1 - P(X \leq 5) = 1 - 0.6230 \approx 0.377$

c $P(X \leq 5) \approx 0.623$

2 a $X \sim B(5, 0.6)$

$P(X = 2) = 5C2 (0.6)^2 (0.4)^3 \approx 0.230$

b $X \sim B(7, 0.6)$

$P(X \geq 3) = 1 - P(X \leq 2) \approx 0.904$

c $X \sim B(9, 0.6)$

$P(X \geq 5) \approx 0.733$

3 $X \sim B(8, 0.01)$

a i $P(X \geq 1) = 1 - P(X = 0) = 1 - (0.99)^8 = 0.077$

ii $P(X \leq 2) = P(0) + P(1) + P(2)$

$= (0.99)^8 + 8 \times (0.99)^7 \times (0.01) + 28 \times (0.99)^6 \times (0.01)^2$
 $\approx 0.9999 \quad (4 \text{ sf})$

b $P(X = 2 | X \geq 1) = \frac{P(X = 2 \text{ and } X \geq 1)}{P(X \geq 1)}$
 $= \frac{P(X = 2)}{P(X \geq 2)} = \frac{28 \times 0.99^6 \times 0.01^2}{0.077} \approx 0.034$

4 $B(6, 0.35)$

x	0	1	2	3	4	5	6
P(x)	0.0754	0.2437	0.328	0.2355	0.0951	0.0205	0.0018

a Mode $a = 2$

b Median $b = 2$

c $P(X < 4 | X > 2) = \frac{P(X < 4 \text{ and } X > 2)}{P(X > 2)}$
 $= \frac{P(X = 3)}{1 - P(0) - P(1) - P(2)} = \frac{0.2355}{0.3529} \approx 0.667$

5 a

P(X ≤ x)	x = 0	x = 1	x = 2	x = 3	x = 4	x = 5	x = 6	x = 7	x = 8	x = 9	x = 10
n = 2	0.36	0.84	1								
n = 5	0.07776	0.33696	0.68256	0.91296	0.98976	1					
n = 10	0.00605 (3 sf)	0.0464 (3 sf)	0.167 (3 sf)	0.382 (3 sf)	0.633 (3 sf)	0.834 (3 sf)	0.945 (3 sf)	0.98771 (3 sf)	0.99832 (3 sf)	0.99990 (3 sf)	1 (3 sf)

b 26

6 a $X \sim B(n, 0.3)$

$P(X > 3) > 0.7 \Rightarrow P(X \leq 3) < 0.3 \Rightarrow n = 15$

7 a $(0.45)^7 \approx 0.00374$

b $X \sim B(7, 0.45)$

$P(X \geq 2) = 1 - P(X \leq 1)$
 $= 1 - 0.1024 \approx 0.8976$

c $5 \times (0.45)^3 \times (0.55)^4 \approx 0.0417$, as there are only 5 sequences where it rains on 3 consecutive days

8 $X \sim B(n, 0.6)$

Need $P(X \geq 1) > 0.95 \Rightarrow P(X \leq 0) < 0.05$

$\Rightarrow (0.4)^n < 0.05 \Rightarrow n = 4$

Exercise 10E

1 $X \sim B(8, 0.4)$

a $P(X = 5) = 8C5 (0.4)^5 (0.6)^3 \approx 0.124$

b $P(X \leq 5) \approx 0.950$

c $P(X < 5) \approx 0.826$

d Mean $= E(X) = np = 8 \times 0.4 = 3.2$

e Variance $= npq = 8 \times 0.4 \times 0.6 = 1.92$

2 a $P(Y = 1) + P(Y = 2) = P(Y \leq 2) - P(Y \leq 0)$

$= 0.6471 - 0.0824$

≈ 0.565

b $P(Y \leq 2) \approx 0.647$

c $P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - 0.3294 \approx 0.671$

d Median = 2 (first value of x for which CDF ≥ 0.5)

x	0	1	2	3	4	5
f(x) = P(X=x)	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$
F(x) = P(X ≤ x)	$\frac{1}{32}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{13}{16}$	$\frac{31}{32}$	1

b Mode of $T = 2$ or 3 (bimodal)

c Median $= \frac{2+3}{2} = 2.5$

4 a $30C3 (0.02)^3 (0.98)^{27} \approx 0.0188$

b $(0.98)^5 \approx 0.9039$

c Mean $= E(X) = np = 30 \times 0.02 = 0.6$

5 $np = 2$ (1)

$np = 1.5$ (2)

$(2) \div (1) \Rightarrow 1 - p = \frac{1.5}{2} = \frac{3}{4}$

$\Rightarrow p = \frac{1}{4}$ and $n = 8$

6 a i $\left(\frac{3}{4}\right)^{20} \approx 0.0032$

ii $X \sim B\left(20, \frac{1}{4}\right)$

$P(X > 10) = 1 - P(X \leq 10) = 1 - 0.9961 \approx 0.0039$

iii $P(X \leq 5) \approx 0.6172$

b $E(X) = np = 20 \times \frac{1}{4} = 5$

Standard deviation $= \sqrt{\text{Var}(X)} = \sqrt{npq}$

$= \sqrt{20 \times \frac{1}{4} \times \frac{3}{4}} = \frac{1}{4} \sqrt{60} \approx 1.94$

c $Y \sim B(5, 0.0039)$

$P(Y \geq 2) = 1 - P(0) - P(1) = 1 - (0.9961)^5$
 $- 5 \times (0.9961)^4 \times 0.0039 \approx 0.0002$

7 $X \sim B(10, 0.18)$

a $P(X = 2) = {}^{10}C_2 (0.18)^2 (0.82)^8 \approx 0.298$

b $P(X \geq 1) = 1 - P(0) = 1 - (0.82)^{10} \approx 0.863$

c Most likely value = mode = 1

d $Y \sim B(25, 0.18)$

$\therefore E(Y) = np = 25 \times 0.18 = 4.5$

e $\text{Var}(Y) = npq = 25 \times 0.18 \times 0.82 = 3.69$

f Require $P(Z \geq 2) > 0.95$ where $Z \sim B(n, 0.18)$

i.e. $1 - P(0) - P(1) > 0.95$

$\Rightarrow 1 - (0.82)^n - n \times (0.82)^{n-1} \times 0.18 > 0.95$

$\Rightarrow (0.82)^n + n \times (0.82)^{n-1} \times 0.18 < 0.05$

$\Rightarrow (0.82)^{n-1} (0.82 + 0.18n) < 0.05$

$\Rightarrow n = 25$

g Men: $M \sim B(5, 0.22)$

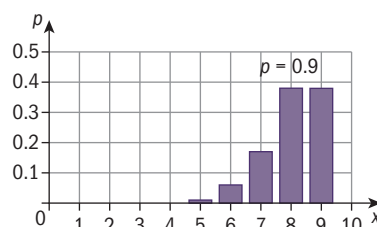
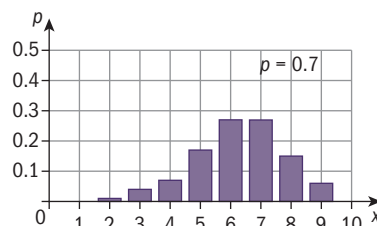
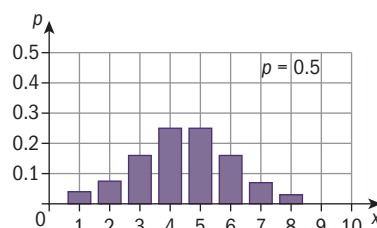
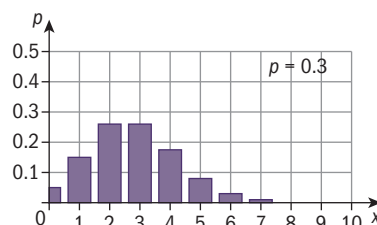
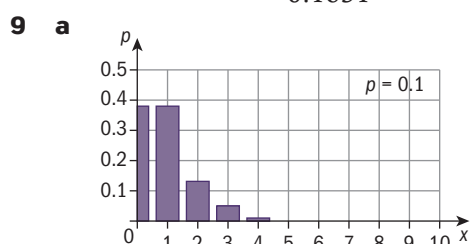
Women: $W \sim B(5, 0.16)$

$P(W \geq 1) \times P(M \geq 1) = [1 - P(W = 0)] \times [1 - P(M = 0)]$
 $= [1 - (0.84)^5] \times [1 - (0.78)^5]$
 $= 0.414$

8 a $X \sim B(5, 0.3)$

i $P(X = 4) = {}^5C_4 \times (0.3)^4 \times 0.7 \approx 0.0567$

ii $P(X \geq 3) = 1 - P(X \leq 2) = 1 - 0.0369$
 $= 0.1631$



- b The graph is symmetrical when $p = 0.5$ and asymmetrical with respect to the line $x = 4.5$ otherwise. For $p < 0.5$ the graph is positively skewed and for $p > 0.5$ it is negatively skewed. The graphs for values of p that add up to 1 are reflections of each other in the line $x = 4.5$

c

	mean	mode	median
$p = 0.1$	0.9	0 and 1	1
$p = 0.3$	2.7	2 and 3	3
$p = 0.5$	4.5	4 and 5	4.5
$p = 0.7$	6.3	6 and 7	6
$p = 0.9$	8.1	8 and 9	8

The values of the parameters of the distributions reflect the symmetries observed (eg. the sum of the means for $p = 0.1$ and $p = 0.9$ is 9).

10 a

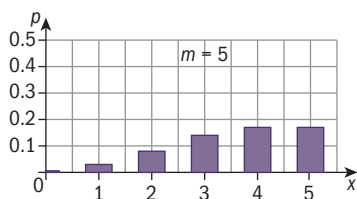
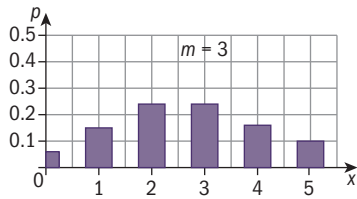
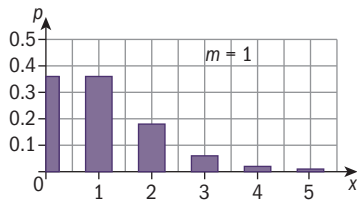
w	0	1	2	4
$P(W = w)$	$(1-a)^2(1-2b)$	$4ab$ $(1-a)(1-b)$	$2ab$ $(a+b-2ab)$	a^2b^2

b $E(W) = \sum x P(W = x) = 4ab(1-a)(1-b)$
 $+ 4ab(a+b-2ab)$
 $+ 4a^2b^2$
 $= 4ab(1-a-b+ab+a+b-2ab) + 4a^2b^2$
 $= 4ab(1-ab+ab) = 4ab$

Exercise 10F

$P(X=x)$	0	1	2	3	4	5
$m=1$	0.368	0.368	0.184	0.0613	0.0153	0.00307
$m=3$	0.0498	0.149	0.224	0.224	0.168	0.101
$m=5$	0.0067	0.0337	0.0842	0.140	0.176	0.176

(to 3 sf)



- 2 a $P(Y=3) = \frac{e^{-3}3^3}{3!} \approx 0.224$
 b $P(Y < 3) = P(0) + P(1) + P(2) \approx 0.423$
 c $P(Y > 3) = 1 - P(Y \leq 3) = 1 - 0.6472 \approx 0.353$
 d $P(Y=4|Y > 3) = \frac{P(Y=4 \text{ and } Y > 3)}{P(Y > 3)} = \frac{P(Y=4)}{P(Y > 3)} \approx 0.476$

Exercise 10G

- 1 a $P(X=2) = \frac{e^{-m}m^2}{2!} = \frac{e^{-0.7} \times 0.49}{2} \approx 0.122$
 b $P(X \geq 2) = 1 - P(0) - P(1) = 1 - e^{-0.7} \times 0.7 = 1 - 1.7e^{-0.7} \approx 0.156$
 2 a $P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.6472 \approx 0.353$
 b $P(P \leq 2) \approx 0.423$

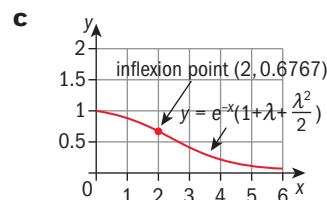
Exercise 10H

- 1 a $X \sim P_0(7), P(X \leq 6) = 0.4497$
 b $Y \sim P_0(1.75), P(Y \geq 2) = 1 - P(0) - P(1) = 1 - e^{-1.75} - e^{-1.75} \times 1.75 = 1 - 2.75e^{-1.75} \approx 0.5221$
 2 a $X \sim P_0(1)$
 $P(X=2) = \frac{e^{-m}m^2}{2!} = \frac{e^{-1}}{2} \approx 0.184$
 b $Y \sim P_0(0.25)$
 $P(Y \geq 1) = 1 - P(0) = 1 - e^{-0.25} \approx 0.221$

- 3 a $X \sim P_0(3) \Rightarrow E(X) = 3$
 b $Y \sim P_0(2) \Rightarrow P(Y > 5) = 1 - P(Y \leq 5) = 1 - 0.9834 \approx 0.017$
 4 a i $P(X=3) = \frac{e^{-3.5}(3.5)^3}{3!} \approx 0.216$
 ii $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.5366 \approx 0.463$
 iii $P(X < 5 | X > 3) = \frac{P(X < 5 \text{ and } X > 3)}{P(X > 3)} = \frac{P(X=4)}{P(X > 3)} = \frac{0.1888}{0.4634} \approx 0.407$
 b $E(X) = m = 3.5, \text{Var}(X) = m = 3.5$
 c $E(X^2) - (E(X))^2 = \text{Var}(X) \Rightarrow E(X^2) - (3.5)^2 = 3.5 \Rightarrow E(X^2) = 15.75$
 5 a $P(0) + P(1) - P(4) = 0 \Rightarrow e^{-m} + e^{-m}m - \frac{e^{-m}m^4}{4!} = 0 \Rightarrow 1 + m - \frac{m^4}{24} = 0 \Rightarrow m^4 = 24(1+m) \Rightarrow m = 3.2 \text{ (2 sf)}$

- 6 $P(X > 3) = 0.555 \Rightarrow P(X \leq 3) = 0.445 \Rightarrow m \approx 3.94 \text{ (3 sf)} \Rightarrow P(X=3) = \frac{e^{-3.94}(3.94)^3}{3!} \approx 0.198 \Rightarrow P(X < 3) = 0.445 - 0.198 \approx 0.247$

- 7 a $P = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda}$
 b $\frac{dP}{d\lambda} = \frac{-e^{-\lambda}}{2} (2 + 2\lambda + \lambda^2) + \frac{e^{-\lambda}}{2} (2 + 2\lambda) = \frac{e^{-\lambda}}{2} (-2 - 2\lambda - \lambda^2 + 2 + 2\lambda) = \frac{-\lambda^2 e^{-\lambda}}{2}$
 λ^2 is always ≥ 0 and $e^{-\lambda}$ is always > 0 , so $\frac{dP}{d\lambda} \leq 0$.
 Hence $P(\lambda)$ is a decreasing function.



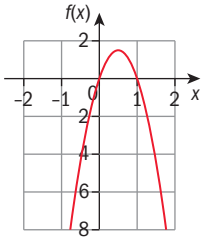
Exercise 10I

- 1 a discrete b continuous
 c continuous (sometimes treated as discrete)
 d discrete as milk is bought in prepacked containers of fixed sizes

2 $f(x) \geq 0$ for all values of x and

$$\int_0^2 f(x) dx = \int_0^2 \frac{1}{2} x dx = \left[\frac{x^2}{2} \right]_0^2 = 1 - 0 = 1$$

3 $f(x)$ has graph



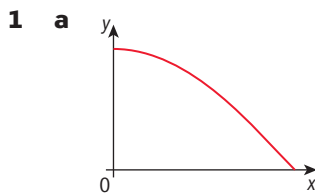
a $E(X) = \frac{1}{2}$ by symmetry

$$\begin{aligned} \text{b } \text{Var}(X) &= E(X^2) - \left(\frac{1}{2}\right)^2 \\ &= \int_0^1 x^2 \cdot 6x(1-x) dx - \frac{1}{4} \\ &= 6 \int_0^1 (x^3 - x^4) dx - \frac{1}{4} \\ &= 6 \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 - \frac{1}{4} \\ &= 6 \times \left(\frac{1}{4} - \frac{1}{5} \right) - \frac{1}{4} \\ &= \frac{6}{20} - \frac{1}{4} = \frac{1}{20} \end{aligned}$$

c Median of $X = \frac{1}{2}$ by symmetry

d Mode of $X = \frac{1}{2}$ by symmetry.

Exercise 10J



b Median M is such that $\int_0^M 2 \cos(2x) dx = \frac{1}{2}$

$$\Rightarrow \sin 2M = \frac{1}{2}$$

$$\Rightarrow 2M = \frac{\pi}{6} \Rightarrow M = \frac{\pi}{12}$$

$$\begin{aligned} \text{Mean} &= \int_0^{\frac{\pi}{4}} x \cdot 2 \cos(2x) dx \\ &= \left[x \sin(2x) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin 2x dx \\ &= \frac{\pi}{4} + \left[\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

Mode of $X = 0$

2 a $a = \int_0^1 x(1-2x+x^2) dx = 1$

$$\Rightarrow a \int_0^1 (x-2x^2+x^3) dx = 1$$

$$\Rightarrow a \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 = 1$$

$$\Rightarrow a \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = 1 \Rightarrow a = 12$$

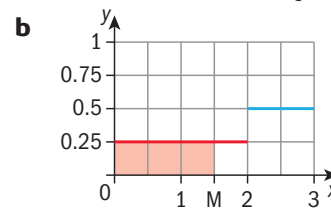
b $E(X) = \int_0^1 x f(x) dx = 12 \int_0^1 x^2(1-2x+x^2) dx$

$$\begin{aligned} &= 12 \int_0^1 (x^2 - 2x^3 + x^4) dx \\ &= 12 \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1 \\ &= 12 \times \frac{1}{30} = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \int_0^1 x^2 f(x) dx - \frac{4}{25} \\ &= 12 \int_0^1 x^3(1-2x+x^2) dx - \frac{4}{25} \\ &= 12 \left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6} \right]_0^1 - \frac{4}{25} \\ &= 12 \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) - \frac{4}{25} = \frac{1}{25} \end{aligned}$$

3 a $\int_0^2 k dx + \int_2^3 2k dx = 1$

$$\Rightarrow 2k + 2k = 1 \Rightarrow k = \frac{1}{4}$$



Median M given by $\int_0^M f(x) dx = \frac{1}{2}$

i.e. shaded area $= \frac{1}{2} \therefore Mk = \frac{1}{2}$

$$M = \frac{1}{2k} = 2$$

c $E(X) = \int_0^3 x f(x) dx = \int_0^2 kx dx + \int_2^3 2kx dx$

$$\begin{aligned} &= \left[\frac{kx^2}{2} \right]_0^2 + \left[kx^2 \right]_2^3 \\ &= 2k + k(9-4) = 7k = \frac{7}{4} = 1\frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_0^3 x^2 f(x) dx - \frac{49}{16} \\ &= \int_0^2 kx^2 dx + \int_2^3 2kx^2 dx - \frac{49}{16} \\ &= \left[\frac{kx^3}{3} \right]_0^2 + \left[\frac{2kx^3}{3} \right]_2^3 - \frac{49}{16} \\ &= \frac{37}{48} \end{aligned}$$

4 a $\int_0^3 (ax^2 + b) dx = 1$

$$\Rightarrow \left[\frac{ax^3}{3} + bx \right]_0^3 = 1 \Rightarrow 9a + 3b = 1$$

$$\Rightarrow 3b = 1 - 9a \Rightarrow b = \frac{1-9a}{3}$$

$$\begin{aligned} \text{b } \int_0^M f(x) dx &= \frac{1}{2} \Rightarrow \int_0^1 (ax^2 + b) dx = \frac{1}{2} \\ &\Rightarrow \left[\frac{ax^3}{3} + bx \right]_0^1 = \frac{1}{2} \\ &\Rightarrow \frac{a}{3} + b = \frac{1}{2} \end{aligned}$$

Sub. in for b from part **a**

$$\Rightarrow \frac{a}{3} + \frac{1-9a}{3} = \frac{1}{2}$$

$$\Rightarrow a + 1 - 9a = \frac{3}{2}$$

$$\Rightarrow 1 - \frac{3}{2} = 8a$$

$$\Rightarrow a = -\frac{1}{16}$$

$$\text{and } \therefore b = \frac{1 + \frac{9}{16}}{3} = \frac{25}{48}$$

$$\begin{aligned} \text{c } E(X) &= \int_0^3 x f(x) dx \\ &= \int_0^3 \left(-\frac{1}{16} x^3 + \frac{25}{48} x \right) dx \\ &= \left[-\frac{x^4}{64} + \frac{25x^2}{96} \right]_0^3 \\ &= -\frac{81}{64} + \frac{75}{32} = 1 - \frac{5}{64} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_0^3 x^2 f(x) dx - \left(\frac{69}{64} \right)^2 \\ &= \int_0^3 \left(-\frac{1}{16} x^4 + \frac{25}{48} x^2 \right) dx - \left(\frac{69}{64} \right)^2 \\ &= \left[-\frac{x^5}{80} + \frac{25x^3}{144} \right]_0^3 - \left(\frac{69}{64} \right)^2 \\ &= \left(-\frac{243}{80} + \frac{675}{144} \right) - \left(\frac{69}{64} \right)^2 \approx 0.488 \end{aligned}$$

$$\begin{aligned} \text{5 a } E(T) &= \int_2^3 t f(t) dt \\ &= \int_2^3 \frac{6}{t} dt \\ &= [6 \ln t]_2^3 \\ &= 6 \ln 3 - 6 \ln 2 = 6 \ln \left(\frac{3}{2} \right) \end{aligned}$$

$$\begin{aligned} \text{b } \text{Var}(T) &= \int_2^3 t^2 f(t) dt - \left[6 \ln \left(\frac{3}{2} \right) \right]^2 \\ &= \int_2^3 6 dt - \left[6 \ln \left(\frac{3}{2} \right) \right]^2 \\ &= 6 - 36 \left[\ln \left(\frac{3}{2} \right) \right]^2 \end{aligned}$$

$$\begin{aligned} \text{c } M \text{ is such that } \int_2^M \frac{6}{t^2} dt &= \frac{1}{2} \\ \Rightarrow \left[-\frac{6}{t} \right]_2^M &= \frac{1}{2} \\ \Rightarrow -\frac{6}{M} + 3 &= \frac{1}{2} \\ \Rightarrow \frac{6}{M} &= \frac{5}{2} \\ \Rightarrow M &= \frac{2 \times 6}{5} = \frac{12}{5} = 2\frac{2}{5} \end{aligned}$$

d Mode of T occurs where $f(t)$ is greatest

$$f'(t) = \frac{-12}{t^3} = \text{negative for } 2 \leq t \leq 3$$

$\therefore f(t)$ is decreasing for $2 \leq t \leq 3$

Or : Mode = 2

Exercise 10K

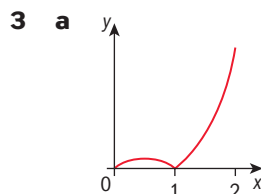
$$\text{1 a } F(x) = \int_0^x \frac{t^2}{9} dt = \left[\frac{t^3}{27} \right]_0^x = \frac{x^3}{27} \quad (0 \leq x \leq 3)$$

$$\text{c } F(x) = \int_0^x \cos t dt = \sin x \quad (0 \leq x \leq \frac{\pi}{2})$$

$$\begin{aligned} \text{2 a } F(x) &= \int_0^x \frac{1}{4} t(4t^2) dt \\ &= \frac{1}{4} \int_0^x (4t - t^3) dt \\ &= \frac{1}{4} \left[2x^2 - \frac{x^4}{4} \right] = \frac{1}{16} (8x^2 - x^4) \quad (0 \leq x \leq 2) \end{aligned}$$

$$\begin{aligned} \text{b } P\left(1 \leq X \leq \frac{3}{2}\right) &= F\left(\frac{3}{2}\right) - F(1) \\ &= \frac{1}{16} \left(8 \times \frac{9}{4} - \frac{81}{16} - 8 + 1 \right) \\ &= \frac{1}{16} \left(10 - \frac{65}{16} \right) = \frac{95}{256} \end{aligned}$$

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - F(1) \\ &= 1 - \frac{1}{16} (8 - 1) \\ &= \frac{9}{16} \end{aligned}$$



$$\begin{aligned} \text{b } k \int_0^2 (x^2 - x) dx &= 1 \\ \Rightarrow k \int_0^1 (x^2 - x) dx + k \int_1^2 (x^2 - x) dx &= 1 \\ \Rightarrow k \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 + k \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 &= 1 \\ \Rightarrow k \left(\frac{1}{3} - \frac{1}{2} \right) + k \left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} \right) &= 1 \\ \Rightarrow k \left(\frac{1}{6} \right) + k \left(\frac{5}{6} \right) &= 1 \Rightarrow k = 1 \end{aligned}$$

$$\begin{aligned} \text{c } P(1 \leq X \leq 2) &= \int_1^2 k |x^2 - x| dx \\ &= \int_1^2 (x^2 - x) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 = \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{7}{3} - \frac{3}{2} = \frac{5}{6} \end{aligned}$$

$$4 \quad a \quad \int_0^1 f(t) dt = 1$$

$$\Rightarrow k \int_0^1 (e - e^{kt}) dt = 1$$

$$\Rightarrow k \left[et - \frac{1}{k} e^{kt} \right]_0^1 = 1$$

$$\Rightarrow k \left[e - \frac{1}{k} e^k + \frac{1}{k} \right] = 1$$

$$\Rightarrow ek - e^k + 1 = 1$$

$$\Rightarrow e^k = ek$$

$$\Rightarrow e^{k-1} = k$$

$$\Rightarrow k = 1 \quad \text{is the solution}$$

$$b \quad P\left(\frac{1}{3} < T < \frac{2}{3}\right) = \int_{\frac{1}{3}}^{\frac{2}{3}} (e - e^t) dt$$

$$= [et - e^t]_{\frac{1}{3}}^{\frac{2}{3}}$$

$$= e^{\frac{2}{3}} - e^{\frac{1}{3}} - e^{\frac{1}{3}} + e^{\frac{1}{3}}$$

$$= \frac{1}{3}e + e^{\frac{1}{3}} - e^{\frac{2}{3}}$$

$$c \quad E(T) = \int_0^1 x f(x) dx$$

$$= \int_0^1 (ex - xe^x) dx = \left[\frac{ex^2}{2} \right]_0^1 - \int_0^1 xe^x dx$$

$$= \frac{1}{2}e - \left[xe^x - \int e^x dx \right]_0^1$$

$$= \frac{1}{2}e - [xe^x - e^x]_0^1$$

$$= \frac{1}{2}e - e + e - 1 = \frac{1}{2}e - 1$$

$$\text{Var}(T) = \int_0^1 x^2 f(x) dx - \left(\frac{1}{2}e - 1\right)^2$$

$$= \int_0^1 x^2 e dx - \int_0^1 x^2 e^x dx - \left(\frac{1}{2}e - 1\right)^2$$

$$= \left[\frac{x^3 e}{3} \right]_0^1 - \left[x^2 e^x - \int 2xe^x dx \right]_0^1 - \left(\frac{1}{2}e - 1\right)^2$$

$$= \frac{e}{3} - [x^2 e^x]_0^1 + 2 \int_0^1 xe^x dx - \left(\frac{1}{2}e - 1\right)^2$$

$$= \frac{e}{3} - e + 2 \left[xe^x - \int e^x dx \right]_0^1 - \left(\frac{1}{2}e - 1\right)^2$$

$$= \frac{2}{3}e + 2[xe^x - e^x]_0^1 - \left(\frac{1}{2}e - 1\right)^2$$

$$= \frac{2}{3}e + 2 - \left(\frac{1}{2}e - 1\right)^2$$

$$= \frac{2}{3}e + 2 - \frac{1}{4}e^2 + e - 1$$

$$= 1 + \frac{1}{3}e - \frac{1}{4}e^2$$

$$d \quad \text{Prob.} = \int_{\frac{1}{2}}^1 (e - e^t) dt = [et - e^t]_{\frac{1}{2}}^1$$

$$= (e - e) - \left(\frac{1}{2}e - e^{\frac{1}{2}}\right)$$

$$= \sqrt{e} - \frac{1}{2}e \approx 0.29$$

$$5 \quad a \quad \int_0^2 \frac{x}{4} dx + \int_2^a \frac{5}{x^2} dx = 1$$

$$\Rightarrow \left[\frac{x^2}{8} \right]_0^2 - \left[\frac{5}{x} \right]_2^a = 1$$

$$\Rightarrow \frac{1}{2} - \frac{5}{a} + \frac{5}{2} = 1$$

$$\Rightarrow 3 = 1 + \frac{5}{a}$$

$$\Rightarrow \frac{5}{a} = 2 \Rightarrow a = \frac{5}{2}$$

b CDF:

$$\text{For } 0 \leq x \leq 2, F(x) = \int_0^x \frac{t}{4} dt = \frac{x^2}{8}$$

$$\text{For } 2 \leq x \leq \frac{5}{2}, F(x) = \int_0^2 \frac{t}{4} dt + \int_2^x \frac{5}{t^2} dt$$

$$= \frac{1}{2} - \left[\frac{5}{t} \right]_2^x$$

$$= \frac{1}{2} - \frac{5}{x} + \frac{5}{2}$$

$$= 3 - \frac{5}{x}$$

$$6 \quad a \quad \int_{-3}^3 \lambda(y+3) dy = 1$$

$$\Rightarrow \lambda \left[\frac{y^2}{2} + 3y \right]_{-3}^3 = 1$$

$$\Rightarrow \lambda \left[\frac{9}{2} + 9 - \frac{9}{2} + 9 \right] = 1$$

$$\Rightarrow 18\lambda = 1 \quad \text{or } \lambda = \frac{1}{18}$$

$$b \quad f(y) = \int_{-3}^y \lambda(x+3) dx$$

$$= \frac{1}{18} \left[\frac{x^2}{2} + 3x \right]_{-3}^y$$

$$= \frac{1}{18} \left(\frac{y^2}{2} + 3y - \frac{9}{2} + 9 \right)$$

$$= \frac{1}{36} (y^2 + 6y + 9) = \frac{1}{36} (y+3)^2$$

$$c \quad P(0 \leq Y \leq 1) = F(1) - F(0) = \frac{16}{36} - \frac{9}{36} = \frac{7}{36}$$

$$P(Y > 1) = 1 - P(Y \leq 1) = 1 - F(1) = 1 - \frac{16}{36} = \frac{5}{9}$$

$$d \quad E(Y) = \int_{-3}^3 y f(y) dy$$

$$= \frac{1}{18} \int_{-3}^3 (y^2 + 3y) dy = \frac{1}{18} \left[\frac{y^3}{3} + \frac{3y^2}{2} \right]_{-3}^3$$

$$= \frac{1}{18} \left[9 + \frac{27}{2} + 9 - \frac{27}{2} \right] = 1$$

$$E(Y^2) = \frac{1}{18} \int_{-3}^3 (y^3 + 3y^2) dy$$

$$= \frac{1}{18} \left[\frac{y^4}{4} + y^3 \right]_{-3}^3$$

$$= \frac{1}{18} \left[\frac{81}{4} + 27 - \frac{81}{4} + 27 \right]$$

$$= \frac{54}{18} = 3$$

$$\therefore \text{Var}(Y) = 3 - 1^2 = 2$$

$$\begin{aligned} \text{e } \frac{1}{36}(y+3)^2 &= \frac{1}{4} \\ \Rightarrow (y+3)^2 &= 9 \\ \Rightarrow y+3 &= \pm 3 \\ \Rightarrow y &= 0 \text{ or } -6 \text{ (impossible)} \\ \therefore y &= 0 \text{ (the lower quartile).} \end{aligned}$$

$$\begin{aligned} 7 \quad \int_1^4 (ax^2 + bx + c) dx &= 1 \\ \Rightarrow \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_1^4 &= 1 \\ \Rightarrow a \frac{64-1}{3} + b \frac{16-1}{2} + c \times 3 &= 1 \\ \Rightarrow 21a + \frac{15b}{2} + 3c &= 1 \quad (1) \end{aligned}$$

$$\begin{aligned} \text{Mode} = 2 &\Rightarrow 2ax + b = 0 \text{ when } x = 2 \\ \Rightarrow 4a + b &= 0 \end{aligned}$$

$$\Rightarrow b = -4a \quad (2)$$

$$E(X) = 3 \Rightarrow \int_1^4 (ax^3 + bx^2 + cx) dx = 3$$

$$\begin{aligned} \Rightarrow \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} \right]_1^4 &= 3 \\ \Rightarrow 64a + \frac{64b}{3} + 8c - \frac{a}{4} - \frac{b}{3} - \frac{c}{2} &= 3 \end{aligned}$$

$$\Rightarrow \frac{255a}{4} + \frac{63b}{3} + \frac{15c}{2} = 3$$

$$\Rightarrow \frac{255a}{4} + 21b + \frac{15c}{2} = 3$$

$$\Rightarrow 255a + 84b + 30c = 12 \quad (3)$$

$$\text{Sub (2) into (1)} \Rightarrow 21a - \frac{60a}{2} + 3c = 1$$

$$\Rightarrow -9a + 3c = 1 \quad (4)$$

$$\text{Sub (2) into (3)}$$

$$\Rightarrow 255a - 336a + 30c = 12$$

$$\begin{array}{rcl} a \times 4 & \Rightarrow & -81a + 30c = 12 \\ & & -81a + 27c = 9 \\ \hline & & 3c = 3 \end{array}$$

$$c = 1 \Rightarrow a = \frac{2}{9} \text{ from (4)}$$

$$\therefore b = -\frac{8}{9} \text{ from (2)}$$

$$8 \quad \text{a } \alpha \int_{\frac{\pi}{2}}^{\frac{\sqrt{\pi}}{2}} (1 + \sin x) dx = 1 \Rightarrow \alpha \left[x - \cos x \right]_{\frac{\pi}{2}}^{\frac{\sqrt{\pi}}{2}} = 1$$

$$\Rightarrow \alpha \left(\frac{\sqrt{\pi}}{2} - 0 - \frac{\pi}{2} + 0 \right) = 1$$

$$\Rightarrow 2\pi\alpha = 1 \Rightarrow \alpha = \frac{1}{2\pi}$$

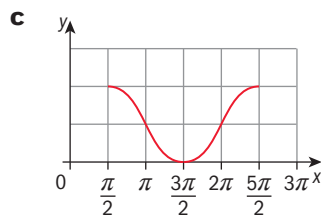
$$\text{b } P(X < \pi) = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2\pi} (1 + \sin x) dx$$

$$= \frac{1}{2\pi} [x - \cos x]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2\pi} \left[\pi + 1 - \frac{\pi}{2} + 0 \right]$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{2} + 1 \right] = \frac{1}{4} + \frac{1}{2\pi}$$

$$\begin{aligned} P(X < 2\pi) &= \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2\pi} (1 + \sin x) dx \\ &= \frac{1}{2\pi} \left[2\pi - \cos 2\pi - \frac{\pi}{2} + 0 \right] \\ &= \frac{1}{2\pi} \left[\frac{3\pi}{2} - 1 \right] \\ &= \frac{3}{4} - \frac{1}{2\pi} \end{aligned}$$



$$\text{Median} = \frac{3\pi}{2}$$

$$\begin{aligned} \text{d } F(x) &= \int_{\frac{\pi}{2}}^x f(t) dt \\ &= \frac{1}{2\pi} [t - \cos t]_{\frac{\pi}{2}}^x \\ &= \frac{1}{2\pi} \left[x - \cos x - \frac{\pi}{2} + 0 \right] \\ &= \frac{x}{2\pi} - \frac{1}{2\pi} \cos x - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{e } F(a) &= 0.75 \\ \Rightarrow \frac{a}{2\pi} - \frac{1}{2\pi} \cos a - \frac{1}{4} &= \frac{3}{4} \end{aligned}$$

$$\Rightarrow a - \cos a = 2\pi$$

$$\Rightarrow a = 7.0 \text{ (1 dp)}$$

$$\text{Similarly for } b, \quad F(b) = 0.25$$

$$\Rightarrow b - \cos b = \pi$$

$$b = 2.4 \text{ (1 dp)}$$

$$\Rightarrow \text{IQR} = a - b \approx 4.6$$

Exercise 10L

$$\begin{aligned} 1 \quad \text{a } P(0 < X < 1.5) &= P\left(\frac{0-1}{2} < Z < \frac{1.5-1}{2}\right) \\ &= F(0.25) - F(-0.5) \\ &= F(0.25) - (1 - F(0.5)) \\ &= 0.5987 + 0.6915 - 1 \\ &\approx 0.290 \end{aligned}$$

$$\begin{aligned} \text{b } P(X < 0.5) &= P\left(Z < \frac{0.5-1}{2}\right) \\ &= P\left(Z < -\frac{1}{4}\right) = F(-0.25) \\ &= 1 - F(0.25) \approx 0.401 \end{aligned}$$

$$\begin{aligned} \text{c } P(X \geq 3) &= P\left(Z \geq \frac{3-1}{2}\right) \\ &= P(Z \geq 1) \\ &= 1 - F(1) \approx 0.159 \end{aligned}$$

$$\begin{aligned} 2 \quad \text{a } P(X < 45) &= P\left(Z < \frac{45-50}{20}\right) \\ &= F(-0.25) \\ &= 1 - F(0.25) \approx 0.401 \end{aligned}$$

- b** $P(37 \leq X < 65) = P\left(\frac{37-50}{20} \leq Z < \frac{65-50}{20}\right)$
 $= F(0.75) - F(-0.65) = F(0.75) - 1 + F(0.65)$
 ≈ 0.516
- c** $P(X \geq 52) = P\left(Z \geq \frac{52-50}{20}\right) = P(Z \geq 0.1)$
 $= 1 - F(0.1)$
 ≈ 0.460
- 3 a** Mean = 35, SD = 7
- b** $P(X < 25) = P\left(Z < \frac{25-35}{7}\right) \approx P(Z < -1.43)$
 $= 1 - F(1.43) \approx 0.076$
 $P(29 \leq X \leq 41) = P\left(\frac{29-35}{7} \leq Z \leq \frac{41-35}{7}\right)$
 $\approx P(-0.86 \leq Z \leq 0.86)$
 $= F(0.86) - F(-0.86)$
 $= F(0.86) - 1 + F(0.86)$
 ≈ 0.610
 $P(X \geq 45) = P\left(Z \geq \frac{45-35}{7}\right)$
 $\approx P(Z \geq 1.43)$
 $= 1 - F(1.43)$
 ≈ 0.076

Exercise 10M

- 1** $X \sim N(150, 0.5^2)$
- a** $P(X < 149) = P\left(Z < \frac{149-150}{0.5}\right)$
 $= P(Z > -2)$
 $= 1 - P(Z < 2)$
 ≈ 0.023
- b** $P(X > 151.5) = P\left(Z < \frac{151.5-150}{0.5}\right)$
 $= P(Z < -3)$
 $= 1 - P(Z < 3)$
 ≈ 0.001
- c** $P(149 < X < 151) = P(-2 < Z < 2)$
 $= F(2) - F(-2)$
 $= F(2) - (1 - F(2))$
 $= 2F(2) - 1$
 ≈ 0.9544
- 2 a** $M \sim N(1.1, 0.15^2)$ ($\mu = 1.1, \sigma = 0.15$)
- b** $P(1.2 < M < 1.3) = P\left(\frac{1.2-1.1}{0.15} < Z < \frac{1.3-1.1}{0.15}\right)$
 $= F(1.33) - F(0.67)$
 $= 0.9082 - 0.7486$
 ≈ 0.160
- c** $P(M > 1.4) = P\left(Z > \frac{1.4-1.1}{0.15}\right) = P(Z > 2)$
 $= 1 - F(2) = 0.0228$
 Estimated no. of cabbages = $0.0228 \times 850 \approx 19$

- 3** $T \sim N(20.4, 3.5^2)$
- a** $P(T < 18.1) = P\left(Z < \frac{18.1-20.4}{3.5}\right) = P(Z < -0.66)$
 $= 1 - F(0.66)$
 ≈ 0.255
 $P(T > 17.9) = P\left(Z > \frac{17.9-20.4}{3.5}\right) = P(Z > -0.71)$
 $= F(0.71)$
 ≈ 0.761
- b** $P(T < 18.1 | T > 17.9) = \frac{P(17.9 < T < 18.1)}{P(T > 17.9)}$
 $= \frac{0.7611 - 0.2546}{0.7611}$
 ≈ 0.665
- c** $P(T < t) = 0.444$
 $\Rightarrow P\left(Z < \frac{t-20.4}{3.5}\right) = 0.444$
 $\Rightarrow F\left(\frac{t-20.4}{3.5}\right) = 0.444$
 $\Rightarrow \frac{t-20.4}{3.5} = -0.14$
 $\Rightarrow t = 20.4 - 0.14 \times 3.5$
 ≈ 19.9

Exercise 10N

- 1** $X \sim N(5, \sigma^2)$ $P(X < 3) = 0.3$
- a** $P(X \geq 7) = P(X \leq 3) = 0.3$
- b** $P(X < 7) = 1 - 0.3 = 0.7$
- c** $P(3 \leq X \leq 7) = 0.7 - 0.3 = 0.4$
- 2** $Y \sim N(12, \sigma^2)$ $P(10 \leq Y < 14) = 0.6$
- a** $P(Y \geq 14) = \frac{1}{2}(1 - 0.6) = 0.2$
- b** $P(Y < 10) = 0.2$
- c** $P(12 \leq Y < 14) = 0.8 - 0.5 = 0.3$
- d** $P(Y < 14 | Y > 12) = \frac{P(12 < Y < 14)}{P(Y > 12)} = \frac{0.3}{0.5} = 0.6$
- 3** $X \sim N(-5, \sigma^2)$ $P(X < -3) = 0.8$
- a** $P(X < -7) = 0.2$
- b** $P(-7 \leq X < -5) = 0.5 - 0.2 = 0.3$
- c** $P(X < -7) + P(X > -3) = 0.2 + 0.2 = 0.4$
- 4 a** Mean = 10, SD = 5
- b** $P(X < 5) = P\left(Z < \frac{5-10}{5}\right) = P(Z < -1)$
 $= 1 - F(1) \approx 0.159$
 $P(X \geq 15) = P\left(Z > \frac{15-10}{5}\right) = P(Z > 1) = 0.159$
- c** $F(a) = 0.223 \Rightarrow \frac{a-10}{5} = -0.76$
 $\Rightarrow a = 10 - 5 \times 0.76 \approx 6.2$
 $\Rightarrow b = 10 + (10 - 6.2) = 13.8$

Exercise 100

- 1 a 0.1151, 0.8849 b 0.0107, 0.9893
c 0.0047, 0.9953

They add to 1 in each case

- 2 a -0.525, 0.525 b -0.253, 0.253

They are equidistant either side of the mean

- 3 a 0.6736, -0.124 b $a = 0$, $b = 0.6915$

- 4 a They are both the same and equal 0.8413

b $P(0 \leq X \leq 2) = P\left(-\frac{1}{2} \leq Z \leq \frac{1}{2}\right) = 0.383$

- 5 $X \sim N(-6, 3^2)$ $Z \sim N(0, 1)$

$$P(0 \leq Z \leq 1.5) = \Phi(1.5) - \Phi(0) = 0.9332 - 0.5 = 0.4332$$

$$P(-6 \leq X \leq -1.5) = P(0 \leq Z \leq 1.5) = 0.4332$$

Exercise 10P

- 1 $P\left(Z < \frac{5-\mu}{9}\right) = 0.754$

$$\Rightarrow \frac{5-\mu}{9} \approx 0.69 \Rightarrow \mu \approx -1.2$$

- 2 $P\left(Z \leq \frac{1-\mu}{\sigma}\right) = 0.345$ and $P\left(Z \leq \frac{3-\mu}{\sigma}\right) = 0.943$

$$\Rightarrow \frac{1-\mu}{\sigma} \approx -0.40 \quad \text{and} \quad \frac{3-\mu}{\sigma} \approx 1.58$$

$$\Rightarrow 1 = \mu - 0.40\sigma \quad 1$$

$$3 = \mu + 1.58\sigma \quad 2$$

$$\Rightarrow 2 = 1.98\sigma \Rightarrow \sigma \approx 1.0 \quad \text{and} \quad \mu \approx 1.40$$

- 3 $P(X > 58.44) = 0.022 \Rightarrow P(X \leq 58.44) = 0.978$

$$\Rightarrow \Phi\left(\frac{58.44 - M}{\sigma}\right) = 0.978$$

$$\Rightarrow \frac{58.44 - M}{\sigma} \approx 2.01$$

$$\Rightarrow \mu = 58.44 - 2.01\sigma \quad (1)$$

$$P(X < 48.84) = 0.012 \Rightarrow \Phi\left(\frac{48.84 - \mu}{\sigma}\right) = 0.012$$

$$\Rightarrow \frac{48.84 - \mu}{\sigma} = -2.26$$

$$\Rightarrow 48.84 + 2.26\sigma = \mu \quad (2)$$

$$\Rightarrow 48.84 + 2.26\sigma = 58.44 - 2.01\sigma \quad (5)$$

$$\Rightarrow 4.27\sigma = 9.6$$

$$\Rightarrow \sigma \approx 2.25$$

Hence $\mu \approx 53.9$

- 4 $X \sim N(1.03, \sigma^2)$

$$\Rightarrow P(X < 1) = 0.018$$

$$\Phi\left(\frac{1-1.03}{\sigma}\right) = 0.018$$

$$\Rightarrow \frac{1-1.03}{\sigma} = -2.10$$

$$\Rightarrow \sigma = \frac{0.03}{2.10} \approx 0.0143 \text{ kg} = 14.3 \text{ grams}$$

- 5 a $\Phi\left(\frac{50.1-\mu}{\sigma}\right) = 1 - 0.119 = 0.881$

$$\Rightarrow \frac{50.1-\mu}{\sigma} = 1.18 \Rightarrow 50.1 = \mu + 1.18\sigma$$

$$\Phi\left(\frac{43.6-\mu}{\sigma}\right) = 0.305 \Rightarrow \frac{43.6-\mu}{\sigma} = -0.51 \quad (1)$$

$$\Rightarrow 43.6 = \mu - 0.51\sigma \quad (2)$$

$$(1) - (2) \Rightarrow 6.5 = 1.69\sigma \Rightarrow \sigma = 3.85 \text{ or } \mu \approx 45.6$$

- b $P(|X - \mu| < \frac{\sigma}{2})$

$$= P\left(-\frac{\sigma}{2} < X - \mu < \frac{\sigma}{2}\right)$$

$$= P\left(-\frac{1}{2} < \frac{X - \mu}{\sigma} < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)$$

$$= 2\Phi\left(\frac{1}{2}\right) - 1$$

$$\approx 0.383$$

- 6 $P(X \geq 80) = 0.1 \Rightarrow P(X < 80) = 0.9$

$$\Rightarrow \Phi\left(\frac{80-\mu}{\sigma}\right) = 0.9$$

$$\Rightarrow \frac{80-\mu}{\sigma} = 1.282$$

$$\Rightarrow 80 = \mu + 1.282\sigma \quad (1)$$

$$P(X < 45) = 0.2$$

$$\Rightarrow \Phi\left(\frac{45-\mu}{\sigma}\right) = 0.2$$

$$\Rightarrow \frac{45-\mu}{\sigma} = -0.842$$

$$\Rightarrow 45 = \mu - 0.842\sigma \quad (2)$$

$$(1) - (2) \Rightarrow 35 = 2.124\sigma$$

$$\Rightarrow \sigma = 16.5$$

$$\mu = 58.9$$

- 7 $X \sim N(550, 20)$

- a $P(500 < X < 600) = P\left(\frac{500-550}{20} < z < \frac{600-550}{20}\right)$

$$= P\left(-2\frac{1}{2} < z < 2\frac{1}{2}\right) = \Phi\left(2\frac{1}{2}\right) - \Phi\left(-2\frac{1}{2}\right)$$

$$= 2\Phi\left(2\frac{1}{2}\right) - 1 \approx 0.988$$

- b $P(X > M) = 0.1 \Rightarrow P(X \leq M) = 0.9$

$$\Rightarrow \Phi\left(\frac{M-550}{20}\right) = 0.9$$

$$\Rightarrow \frac{M-550}{20} = 1.282$$

$$\Rightarrow M = 576 \text{ grams}$$

- c $P(X > 540) = P\left(z > \frac{540-550}{20}\right) = P\left(z > -\frac{1}{2}\right)$

$$= P\left(z < \frac{1}{2}\right) = 0.6915$$

$$\therefore \text{Estimated number} = 1200 \times 0.6915 \approx 830$$

d $M \sim N(\mu, \sigma^2)$

$$P(M \geq 600) = 0.15 \Rightarrow P(M < 600) = 0.85$$

$$\Rightarrow \Phi\left(\frac{600 - \mu}{\sigma}\right) = 0.85$$

$$\frac{600 - \mu}{\sigma} = 1.037$$

$$\Rightarrow 600 = \mu + 1.037\sigma \quad (1)$$

$$P(M < 540) = 0.1$$

$$\Rightarrow \Phi\left(\frac{540 - \mu}{\sigma}\right) = 0.1$$

$$\Rightarrow \frac{540 - \mu}{\sigma} = -1.282$$

$$\Rightarrow 540 = \mu - 1.282\sigma \quad (2)$$

$$(1) - (2) \Rightarrow 60 = 2.319\sigma$$

$$\Rightarrow \sigma = 25.9 \text{ grams}$$

$$\mu = 573 \text{ grams}$$

8 $M \sim N(1.02, \sigma^2)$

$$P(M < 1) < 0.01$$

$$\Rightarrow P\left(Z < \frac{1 - 1.02}{\sigma}\right) < 0.01$$



Review exercise

1 a $\sum P(X) = 1$

$$\Rightarrow 2a^2 + 3a + 3a^2 + 2a + 2a^2 + a = 1$$

$$\Rightarrow 7a^2 + 6a - 1 = 0$$

$$\Rightarrow (7a - 1)(a + 1) = 0$$

$$\Rightarrow a = \frac{1}{7}$$

(a cannot be negative as $P(2)$ would then be negative)

b PDF is: $P(X=x)$

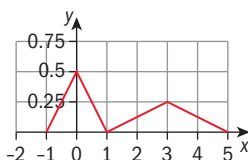
x	1	2	3	4
$\frac{2}{49}$	$\frac{21}{49}$	$\frac{17}{49}$	$\frac{9}{49}$	

$$\begin{aligned} \text{Mean} &= \sum xP(x) = \frac{2}{49} + \frac{42}{49} + \frac{51}{49} + \frac{36}{49} \\ &= \frac{131}{49} = 2\frac{33}{49} \end{aligned}$$

$$\text{Mode} = 2$$

$$\text{Median} = 3$$

2 a



$$\text{Total area} = 1, \text{ so } \frac{1}{2} \times 4 \times \frac{2}{k} + \frac{1}{2} \times 2 \times \frac{1}{2} = 1$$

$$\Rightarrow \frac{4}{k} + \frac{1}{2} = 1 \Rightarrow \frac{4}{k} = \frac{1}{2}$$

$$\Rightarrow k = 8$$

b Median = 1, Mode = 3

c $P(0 \leq X \leq 3 | X \geq 1) = \frac{P(1 \leq X \leq 3)}{P(X \geq 1)}$

$$= \frac{\frac{1}{2} \times 2 \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

3 Let X = number graduating $X \sim B(n, p)$, $n = 5$, $p = \frac{4}{5}$

a $P(X = 0) = \left(\frac{1}{5}\right)^5 = \frac{1}{3125}$

b $P(X = 5) = \left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$

c $P(X \geq 2) = 1 - P(0) - P(1)$

$$= 1 - \left(\frac{1}{5}\right)^5 - 5 \times \left(\frac{1}{5}\right)^4 \times \frac{4}{5}$$

$$= 1 - \frac{21}{3125}$$

$$= \frac{3104}{3125}$$

4 Let X = the no. correct on the last 5 questions

Then $X \sim B(n, p)$ where $n = 5$, $p = \frac{1}{2}$

a $P(X = 5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

b $E(X) = np = 2\frac{1}{2}$

If get X correct Y is the final score, then

$$\begin{aligned} Y &= 5 \times 2 + X \times 2 + (5 - X) \times -1 \\ &= 10 + 2 \times -5 + X \\ &= 5 + 3X \end{aligned}$$

$\therefore$ Expected final score is $E(Y) = 5E(X) + 3 = 5 \times 2.5 + 3 = 12.5$

It is obviously worth guessing the last 5, because if he stops after the first 5, he will only get a score of 10.

5 a $E(T) = \text{Var}(T) = m$

But $\text{Var}(T) = E(T^2) - (E(T))^2$

$$\therefore m = 6 - m^2$$

$$\Rightarrow m^2 + m - 6 = 0$$

$$\Rightarrow (m + 3)(m - 2) = 0$$

$$\Rightarrow m = -3 \text{ or } 2$$

$m = -3$ is impossible as it would give

$$P(0) = e^3 > 1$$

$$\therefore m = 2$$

b $P(X = 0) = \frac{e^{-m} m^0}{0!} = e^{-2} = 0.135 \left(\text{or } \frac{1}{e^2} \right)$

6 a $a = 1$

b $F(m) = \frac{1}{2} \Rightarrow \tan m = \frac{1}{2} \Rightarrow m \approx 0.46 \left(\text{or } \tan^{-1}\left(\frac{1}{2}\right) \right)$

$$\text{c } f(x) = F'(x) = \begin{cases} \sec^2 x & 0 \leq x \leq \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{d } P\left(x \leq \frac{\pi}{4}\right) = F\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$



Review exercise

- 1 Let X = width of a piece

Then $X \sim N(20.05, 0.02^2)$

$$\text{a } \Rightarrow P(20.02 < X < 20.06)$$

$$= P\left(\frac{20.02 - 20.05}{0.02} < z < \frac{20.06 - 20.05}{0.02}\right)$$

$$= P(-1.5 < z < 0.5) = F(0.5) - F(-1.5)$$

$$= 0.691462 - 0.066807$$

$$\approx 0.625$$

$$\text{b } P(X < 20.00) = P\left(z < \frac{20.00 - 20.05}{0.02}\right) = F(-2.5)$$

$$\approx 0.0062$$

2. Let X = life of a motor and G the guarantee period.

Then X is $N(15, 2^2)$ and we require:

$$P(X \leq G) = 0.001$$

$$\text{i.e. } P\left(z \leq \frac{G - 15}{2}\right) = 0.001$$

$$\Rightarrow \frac{G - 15}{2} \approx -3.10$$

$$\Rightarrow G \approx 15 - 6.2$$

$$\approx 8.8$$

(Should only give an 8 year guarantee)

$$\text{3 a } \int_0^{\sqrt{2}} \frac{k}{2+x^2} dx = 1 \Rightarrow k \left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_0^{\sqrt{2}} = 1$$

$$\Rightarrow \frac{k}{\sqrt{2}} \left[\frac{\pi}{4} \right] = 1 \Rightarrow k = \frac{4\sqrt{2}}{\pi} \approx 1.8$$

$$\text{b } P\left(X \leq \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(x) dx = \frac{k}{\sqrt{2}} \left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_0^{\frac{1}{2}}$$

$$= \frac{4}{\pi} \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

$$\approx 0.433$$

$$\text{c } E(X) = \int_0^{\sqrt{2}} x f(x) dx = k \int_0^{\sqrt{2}} \frac{x}{2+x^2} dx$$

$$= k \left[\frac{1}{2} \ln(2+x^2) \right]_0^{\sqrt{2}} = \frac{k}{2} (\ln 2.25 - \ln 2)$$

$$= \frac{k}{2} \ln\left(\frac{2.25}{2}\right)$$

$$\approx 0.106$$

- 4 $X \sim P_0(4)$ ($m = 4$ in 2hrs)

$$\text{a } P(X \geq 1) = 1 - P(0) = 1 - e^{-m}$$

$$\approx 0.982$$

$$\text{b } P(X > 5) = 1 - P(X \leq 5) \approx 0.215$$

$$\text{5 a } a \int_1^3 \frac{1}{x(4-x)} dx = 1$$

$$\Rightarrow \frac{a}{4} \left[\ln\left(\frac{x}{4-x}\right) \right]_1^3 = 1 \Rightarrow \frac{a}{4} \left[\ln 3 - \ln \frac{1}{3} \right] = 1$$

$$\Rightarrow \frac{a}{4} \ln 9 = 1 \Rightarrow a = \frac{4}{\ln 9} \approx 1.82048$$

$$\text{b } E(X) = \int_1^3 x f(x) dx$$

$$= a \int_1^3 \frac{1}{4-x} dx$$

$$= a [-\ln(4-x)]_1^3$$

$$= \frac{4}{\ln 9} \times \ln 3$$

$$= \frac{4}{2} = 2$$

$$\text{Var}(X) = E(X^2) - 2^2$$

$$= \int_1^3 x^2 f(x) dx - 4$$

$$= a \int_1^3 \frac{x^2}{4-x} dx - 4$$

$$= a \int_1^3 -1 + \frac{4}{4-x} dx - 4$$

$$= a [-x - 4 \ln(4-x)]_1^3 - 4$$

$$= a [-3 - 4 \ln 1 + 1 + 4 \ln 3] - 4$$

$$= a [-2 + 4 \ln 3] - 4$$

$$= \frac{4}{\ln 9} \times -2 + \frac{16}{2} - 4$$

$$= \frac{-8}{\ln 9} + 4 \approx 0.35904$$

$$\text{c } P(X < 2) = \int_1^2 f(x) dx = \left[\frac{a}{4} \ln\left(\frac{x}{4-x}\right) \right]_1^2$$

$$= \frac{a}{4} \left(\ln 1 - \ln\left(\frac{1}{3}\right) \right)$$

$$= \frac{1}{\ln 9} \times \ln 3 = \frac{1}{2}$$

$$\text{12 a } 250 \times \frac{71}{160} = 110 \frac{15}{16} \approx 111 \text{ customers}$$

- b i X = no. of customers who buy vanilla

Then $X \sim B\left(5, \frac{3}{16}\right)$

$$\therefore P(X=3) = 5c_3 \left(\frac{3}{16}\right)^3 \left(\frac{13}{16}\right)^2$$

$$\approx 0.0435$$

- ii Y = not buying ice cream.

$Y \sim B\left(5, \frac{3}{160}\right)$

$$P(Y=2) = 5c_2 \left(\frac{3}{160}\right)^2 \left(\frac{157}{160}\right)^3 \approx 0.0033$$

$$\text{c } 2 \times \frac{14}{160} \times \frac{1}{160} \approx 0.0011$$

Have assumed the 2 choices of flavor are independent of one another.

11

Inspiration and formalism

Answers

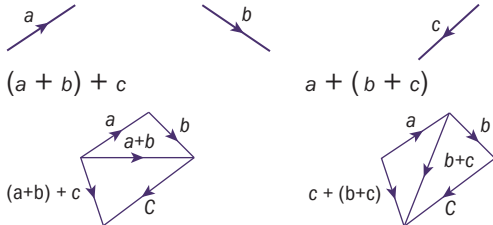
Skills check

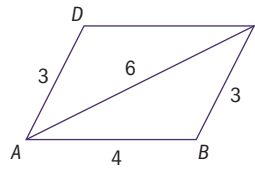
- 1 $P(-1, 5)$ $Q(2, 1)$ $PQ^2 = (-1 - 2)^2 + (5 - 1)^2 = 25$
 $PQ = 5$
- 2 $A(1, 3)$ $B(4, 9)$
- a gradient $= \frac{9-3}{4-1} = 2$
- b $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

Exercise 11A

- 1 a opposite sides of a regular hexagon are equal and parallel $\therefore \overrightarrow{AB} = \overrightarrow{ED}$
- b i $\overrightarrow{FC}$ and $\overrightarrow{ED}$
- ii $\overrightarrow{AD}$, $\overrightarrow{DA}$, $\overrightarrow{BE}$, $\overrightarrow{EB}$ and $\overrightarrow{FC}$
- 2 a No, 2 sides of a pentagon are parallel $\therefore$ the vectors in the diagram are all distinct.
- b No, since no 2 vectors in the diagram are parallel and equal in length

Exercise 11B

- 1
- 
- 2 a $\overrightarrow{AF} + \overrightarrow{BC} = \overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AE}$
- b $\frac{1}{2}\overrightarrow{AD} + \overrightarrow{ED} = \overrightarrow{FE} + \overrightarrow{ED} = \overrightarrow{FD}$
- c $2\overrightarrow{FE} - \overrightarrow{AF} - \overrightarrow{FE} = \overrightarrow{FE} + \overrightarrow{FA}$
 $= \overrightarrow{FE} + \overrightarrow{EO} = \overrightarrow{FO} = \overrightarrow{AB}$
- d $\frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BE}) = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{BE} = \overrightarrow{AO} + \overrightarrow{OE} = \overrightarrow{AE}$
- e $-\frac{1}{2}\overrightarrow{FC} + \overrightarrow{BC} = \overrightarrow{OF} + \overrightarrow{FE} = \overrightarrow{OE} = \overrightarrow{CD}$
- f $-2\overrightarrow{ED} - \overrightarrow{AF} + \overrightarrow{AB} = \overrightarrow{CF} + \overrightarrow{FA} + \overrightarrow{AB} = \overrightarrow{CB}$
 (other answers are possible in this question)
- 3 a i $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$
- ii $\overrightarrow{HB} = \mathbf{u} - \mathbf{w} + \mathbf{u} = 2\mathbf{u} - \mathbf{w}$
- iii $\overrightarrow{CE} = -\mathbf{v} - \mathbf{u} + \mathbf{w} - \mathbf{u} - \mathbf{v} = \mathbf{w} - 2\mathbf{u} - 2\mathbf{v}$
- iv $\overrightarrow{AF} = \mathbf{w} - \mathbf{v}$

b i  $\cos B = \frac{4^2 + 3^2 - 6^2}{2 \times 4 \times 3} = -\frac{11}{24}$
 $\hat{A}BC = 117^\circ$

ii area ABCD $= 2 \left(\frac{1}{2} \times 4 \times 3 \times \sin 117^\circ \right)$
 $= 10.7$ sq. units

- 4 a $3\mathbf{x} - \mathbf{u} = 6\mathbf{v} + 2\mathbf{u} \Rightarrow 3\mathbf{x} = 6\mathbf{v} + 3\mathbf{u}$
 $\Rightarrow \mathbf{x} = 2\mathbf{v} + \mathbf{u}$
- b $2(\mathbf{x} - \mathbf{u}) + 3(\mathbf{u} - \mathbf{v}) = 0 \Rightarrow 2\mathbf{x} - 2\mathbf{u} + 3\mathbf{u} - 3\mathbf{v} = 0$
 $\mathbf{x} = \frac{1}{2}(3\mathbf{v} - \mathbf{u})$
- c $\frac{1}{2}(\mathbf{x} - \mathbf{u}) = \frac{1}{3}(\mathbf{x} + \mathbf{v}) \Rightarrow 3\mathbf{x} - 3\mathbf{u} = 2\mathbf{x} + 2\mathbf{v}$
 $\Rightarrow \mathbf{x} = 3\mathbf{u} + 2\mathbf{v}$

Exercise 11C

- 1 $A(-1, 3)$ $C(5, 4)$ $I(7, 8)$
- a i $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2} \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0.5 \end{pmatrix}$
- ii $\overrightarrow{AE} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{CI}$
 $= \begin{pmatrix} 3 \\ 0.5 \end{pmatrix} + \frac{1}{2} \left[\begin{pmatrix} 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} 4 \\ 2.5 \end{pmatrix}$
- iii $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{CI} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- b i $\overrightarrow{BF} = \overrightarrow{BA} + \overrightarrow{AF} = \begin{pmatrix} -3 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2\mathbf{i} + 1.5\mathbf{j}$
- ii $\overrightarrow{CH} = \overrightarrow{CB} + \overrightarrow{BH} = \begin{pmatrix} -3 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -\mathbf{i} + 3.5\mathbf{j}$
- iii $\overrightarrow{DG} = \overrightarrow{DF} + \overrightarrow{FG} = \begin{pmatrix} -6 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -5\mathbf{i} + \mathbf{j}$
- c $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3.5 \end{pmatrix}$
- $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$
- $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2.5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5.5 \end{pmatrix}$
- $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$
- $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

2 $P(0, 2, -1) Q(2, 1, 1)$

a $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

3 a $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

b $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

c $\overrightarrow{AE} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

d $\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

e $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}$

f $\overrightarrow{BH} = \overrightarrow{BD} + \overrightarrow{DH} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$

4 $P(-3, 1) Q(5, 7) R(-1, 5)$

a $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \overrightarrow{OQ} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \overrightarrow{OR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b $M(1, 4) N(-2, 3)$

c $\overrightarrow{QR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$

$\overrightarrow{MN} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \therefore \overrightarrow{QR} = 2 \overrightarrow{MN} \text{ (QED)}$

Exercise 11D

1 a $\mathbf{u} + (-\mathbf{u}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 - u_1 \\ u_2 - u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$

b $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$
 $= \begin{pmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + w_1 \\ (u_2 + v_2) + w_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$
 $= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \text{ (QED)}$

c $\alpha(\beta\mathbf{u}) = \alpha \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \alpha\beta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\alpha\beta)\mathbf{u}$

$\alpha(\beta\mathbf{u}) = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \begin{pmatrix} \beta\alpha u_1 \\ \beta\alpha u_2 \end{pmatrix} = \beta \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} = \beta(\alpha\mathbf{u})$
 (QED)

d $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_1 + v_1) \\ \alpha(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1 \\ \alpha u_2 + \alpha v_2 \end{pmatrix}$
 $= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha\mathbf{u} + \alpha\mathbf{v} \text{ (QED)}$

e $(\alpha + \beta)\mathbf{u} = (\alpha + \beta) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta)u_1 \\ (\alpha + \beta)u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \beta u_1 \\ \alpha u_2 + \beta u_2 \end{pmatrix}$
 $= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha\mathbf{u} + \beta\mathbf{u} \text{ (QED)}$

f $0\mathbf{u} = 0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0u_1 \\ 0u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$

g $\alpha\mathbf{0} = \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha 0 \\ \alpha 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} \text{ (QED)}$

2 a $2 \begin{pmatrix} x \\ y \end{pmatrix} - 3 \begin{pmatrix} y \\ x \end{pmatrix} = 5 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
 $2x - 3y = 5$

$2y - 3x = -10, \quad x = 4, \quad y = 1$

b $2 \left(\begin{pmatrix} 2 \\ y \end{pmatrix} - \begin{pmatrix} x \\ 2 \end{pmatrix} \right) - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \Rightarrow 2 \begin{pmatrix} 2-x \\ y-2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0$
 $4 - 2x - 1 = 0$
 $2y - 4 - 3 = 0 \quad \therefore x = \frac{3}{2} \quad y = \frac{7}{2}$

3 a $\mathbf{u} + (\mathbf{v} + 2\mathbf{u}) = 3\mathbf{u} + \mathbf{v}$

b $(\mathbf{u} - \mathbf{v}) + 2(\mathbf{v} - 2\mathbf{u}) = \mathbf{u} - \mathbf{v} + 2\mathbf{v} - 4\mathbf{u} = -3\mathbf{u} + \mathbf{v}$

c $3 \left(\frac{1}{6}(\mathbf{u} - \mathbf{v}) + \frac{1}{3}(\mathbf{v} - \mathbf{u}) \right) = \frac{1}{2}(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{u}) = -\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$

4 a $= 2\mathbf{i} - 3\mathbf{j} \quad \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$

$\alpha\mathbf{a} + \beta\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

$\therefore \alpha(2\mathbf{i} - 3\mathbf{j}) + \beta(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$

$\alpha\mathbf{a} + \beta\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

$\therefore \alpha(2\mathbf{i} - 3\mathbf{j}) + \beta(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$

$2\alpha - \beta = 3, -3\alpha - 2\beta = -1 \quad \therefore \alpha = 1 \quad \beta = -1$

$6\mathbf{i} - 2\mathbf{j} = 2\mathbf{a} - 2\mathbf{b}$

Exercise 11E

1 a $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} |\mathbf{v}| = \sqrt{26} \quad \frac{1}{\sqrt{26}} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} |\mathbf{v}| = 13 \quad \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ \frac{12}{13} \end{pmatrix}$

c $\mathbf{v} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} |\mathbf{v}| = 3 \quad \frac{1}{3} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

d $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} |\mathbf{v}| = \sqrt{2} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$2 \quad a \quad \mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{5} \quad \pm \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$b \quad \mathbf{v} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{29} \quad \pm \frac{1}{\sqrt{29}} \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{\sqrt{29}} \\ -\frac{2}{\sqrt{29}} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{5}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \end{pmatrix}$$

$$c \quad \mathbf{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$d \quad \mathbf{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{2} \quad \pm \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$3 \quad \mathbf{u} = 2\mathbf{i} - 3\mathbf{j} \quad |\mathbf{u}| = \sqrt{13} \quad \mathbf{v} = \frac{1}{\sqrt{13}} \mathbf{u} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{pmatrix}$$

$$4 \quad a \quad |\mathbf{u}| = \sqrt{13} \quad \mathbf{v} = \pm \frac{2}{\sqrt{13}} \mathbf{u} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{6}{\sqrt{13}} \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{4}{\sqrt{13}} \\ -\frac{6}{\sqrt{13}} \end{pmatrix}$$

$$b \quad |\mathbf{u}| = 3 \quad \mathbf{v} = \pm \frac{2}{3} \mathbf{u} = \begin{pmatrix} -\frac{4}{3} \\ \frac{2\sqrt{5}}{3} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{4}{3} \\ -\frac{2\sqrt{5}}{3} \end{pmatrix}$$

$$c \quad |\mathbf{u}| = \sqrt{26} \quad \mathbf{v} = \pm \frac{13}{\sqrt{26}} \mathbf{u} = \pm \sqrt{\frac{13}{2}} \mathbf{u} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{2}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{pmatrix}$$

$$5 \quad \mathbf{u} = -4\mathbf{i} - 6\mathbf{j} \quad |\mathbf{u}| = \sqrt{52} = 2\sqrt{13} \\ \mathbf{w} = \frac{1}{2} \mathbf{u} = -2\mathbf{i} - 3\mathbf{j}$$

$$6 \quad \mathbf{u} = \mathbf{i} - 3\mathbf{j} \quad |\mathbf{u}| = \sqrt{10} \quad \mathbf{t} = \pm \frac{5}{\sqrt{10}} \mathbf{u} \\ \mathbf{t} = \frac{5}{\sqrt{10}} \mathbf{i} - \frac{15}{\sqrt{10}} \mathbf{j} \text{ or } \mathbf{t} = -\frac{5}{\sqrt{10}} \mathbf{i} + \frac{15}{\sqrt{10}} \mathbf{j}$$

$$7 \quad \mathbf{u} = \frac{1}{\sqrt{v_1^2 + v_2^2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{v_1^2 + v_2^2}} \mathbf{v} \quad \therefore \mathbf{u} = k\mathbf{v} \quad (k > 0)$$

$\therefore \mathbf{u}$ is in the same direction as $\mathbf{v}$ (QED)

$$|\mathbf{u}|^2 = \frac{v_1^2}{v_1^2 + v_2^2} + \frac{v_2^2}{v_1^2 + v_2^2} = \frac{v_1^2 + v_2^2}{v_1^2 + v_2^2} = 1 \quad \therefore |\mathbf{u}| = 1$$

$\therefore \mathbf{u}$ has magnitude 1 (QED)

$$8 \quad \mathbf{u} = \pm \frac{\mathbf{m}}{|\mathbf{v}|} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \pm \begin{pmatrix} \frac{mv_1}{|\mathbf{v}|} \\ \frac{mv_2}{|\mathbf{v}|} \end{pmatrix}$$

$$|\mathbf{u}|^2 = \frac{m^2 v_1^2}{|\mathbf{v}|^2} + \frac{m^2 v_2^2}{|\mathbf{v}|^2} = \frac{m^2 (v_1^2 + v_2^2)}{v_1^2 + v_2^2} = m^2$$

$\therefore |\mathbf{u}| = m$ (QED)

Exercise 11F

$$1 \quad a \quad A(2, 5) \quad B(5, 6) \quad C(4, 2) \quad D(1, 1)$$

$$b \quad \overrightarrow{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

$$c \quad \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$$2 \quad A(2, 6) \quad B(-2, 4)$$

$$a \quad \overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$$b \quad AB = \sqrt{(-2-2)^2 + (4-6)^2} = \sqrt{20} = 2\sqrt{5} \\ \therefore |\overrightarrow{AB}| = 2\sqrt{5}$$

$$c \quad M(0, 5) \quad M \text{ is the midpoint of } AB$$

$$d \quad \text{Let } P(x_1, y_1) \quad \overrightarrow{AP} = \begin{pmatrix} x_1 - 2 \\ y_1 - 6 \end{pmatrix} \quad \overrightarrow{PB} = \begin{pmatrix} -2 - x_1 \\ 4 - y_1 \end{pmatrix}$$

$$\overrightarrow{AP} = 2\overrightarrow{PB} \quad \therefore x_1 - 2 = 2(-2 - x_1), y_1 - 6 = 2(4 - y_1)$$

$$x_1 - 2 = -4 - 2x_1, y_1 - 6 = 8 - 2y_1$$

$$3x_1 = -2 \quad 3y_1 = 14$$

$$x_1 = -\frac{2}{3} \quad y_1 = \frac{14}{3}$$

$$\therefore P\left(-\frac{2}{3}, \frac{14}{3}\right)$$

$$\text{Let } Q(x_2, y_2), \quad \overrightarrow{AQ} = \begin{pmatrix} x_2 - 2 \\ y_2 - 6 \end{pmatrix} \quad \overrightarrow{QB} = \begin{pmatrix} -2 - x_2 \\ 4 - y_2 \end{pmatrix} \\ \overrightarrow{AQ} = -2\overrightarrow{QB}$$

$$\therefore x_2 - 2 = -2(-2 - x_2), y_2 - 6 = -2(4 - y_2)$$

$$x_2 - 2 = 4 + 2x_2, y_2 - 6 = -8 + 2y_2$$

$$x_2 = -6 \quad y_2 = 2$$

$$\therefore Q(-6, 2)$$

$$\overrightarrow{AB} < \overrightarrow{PQ}$$

$$\overrightarrow{PQ} \text{ has greater magnitude than } \overrightarrow{AB}$$

$$3 \quad P(4, -1) \quad Q(6, -3) \quad R(2, 1)$$

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad \overrightarrow{PR} = -1\overrightarrow{PQ}$$

$$\therefore \overrightarrow{PQ} \text{ and } \overrightarrow{PR} \text{ are collinear}$$

$$\therefore P, Q, R \text{ are collinear (QED)}$$

$$4 \quad A(a, a-1) \quad B(2, 2a) \quad C(0, 3a)$$

$$\overrightarrow{AB} = \begin{pmatrix} 2-a \\ a+1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} -a \\ 2a+1 \end{pmatrix} \quad \overrightarrow{AC} = k\overrightarrow{AB}$$

$$\therefore -a = k(2-a)$$

$$2a+1 = k(a+1) \quad \therefore \frac{-a}{2a+1} = \frac{2-a}{a+1}$$

$$-a^2 - a = 4a + 2 - 2a^2 - a$$

$$a^2 - 4a - 2 = 0$$

$$a = \frac{4 \pm \sqrt{16+8}}{2}, a = 2 \pm \sqrt{6}$$

$$5 \quad S(2, -3) \quad U(-1, 2) \quad N(1, -4)$$

$$\overrightarrow{SU} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad \overrightarrow{SN} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \overrightarrow{SN} \neq k\overrightarrow{SU}$$

$\therefore S, U, N$ are not collinear
 $\therefore$ they form a triangle (QED)

$$6 \quad P(a, b) \quad Q(c, d) \quad R(e, f)$$

$$\overrightarrow{PQ} = \begin{pmatrix} c-a \\ d-b \end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix} e-a \\ f-b \end{pmatrix}$$

$$\frac{f-b}{d-b} = \frac{e-a}{c-a} \Rightarrow e-a = \frac{(f-b)(c-a)}{d-b}$$

$$\therefore \overrightarrow{PR} = \begin{pmatrix} \frac{(f-b)(c-a)}{d-b} \\ \frac{(f-b)(d-b)}{d-b} \end{pmatrix} = \frac{f-b}{d-b} \begin{pmatrix} c-a \\ d-b \end{pmatrix}$$

$\therefore \overrightarrow{PR} = \frac{f-b}{d-b} \overrightarrow{PQ} \quad \therefore \overrightarrow{PR} = k \overrightarrow{PQ}$
 $\therefore P, Q, R$ are collinear points

$$7 \quad a \quad \overrightarrow{AB} = \begin{pmatrix} \sin 2x - \sin x \\ \cos 2x - (-1 + \cos x) \end{pmatrix}$$

$$= \begin{pmatrix} 2\sin x \cos x - \sin x \\ 2\cos^2 x - 1 + 1 - \cos x \end{pmatrix}$$

$$= \begin{pmatrix} \sin x (2\cos x - 1) \\ 2\cos^2 x - \cos x \end{pmatrix}$$

$$= \begin{pmatrix} \sin x (2\cos x - 1) \\ \cos x (2\cos x - 1) \end{pmatrix}$$

$$= (2\cos x - 1) \begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$$

Therefore for any value of x , $\overrightarrow{AB}$ is collinear
 with $\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

$$b \quad |\overrightarrow{AB}| = |2\cos x - 1| \sqrt{\sin^2 x + \cos^2 x}$$

$$= |2\cos x - 1| \sqrt{1}$$

$$= |2\cos x - 1|$$

Exercise 11G

$$1 \quad \mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \quad \mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$a \quad \mathbf{u} + \mathbf{v} = -3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$$

$$b \quad -3\mathbf{u} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$$

$$c \quad 4\mathbf{u} - 2\mathbf{v} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} - (-2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})$$

$$= -6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$$

$$d \quad -2(\mathbf{u} - \mathbf{v}) = -2(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$$

$$2 \quad \mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$a \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix} \quad b \quad 2\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$$

$$c \quad 2(a-b) - 3c = 2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix}$$

$$d \quad \frac{1}{2}(\mathbf{a} - 3\mathbf{b}) = \frac{1}{2} \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$e \quad |\mathbf{a}| = \sqrt{14} \quad f \quad |\mathbf{b}| = 3$$

$$g \quad \mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \quad \therefore |\mathbf{a} + \mathbf{b}| = \sqrt{35}$$

$$h \quad \mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad \therefore |\mathbf{a} - \mathbf{b}| = \sqrt{11}$$

$$3 \quad A(0, 2, 1) \quad B(-1, -1, -2) \quad C(1, -3, 0)$$

$$a \quad \overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$$

$$b \quad \overrightarrow{AB} - \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} \quad \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$4 \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$a \quad \mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad b \quad \pm \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad c \quad \begin{pmatrix} 0 \\ \sqrt{5} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$5 \quad A(4, -1, 3) \quad C(0, -2, 5) \quad D(5, 1, 6) \quad G(1, -4, 6)$$

$$a \quad \overrightarrow{AC} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$b \quad \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{AD} = \begin{pmatrix} -5 \\ -3 \\ -1 \end{pmatrix} \quad \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AE} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$$

$$\overrightarrow{BF} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AH} = \overrightarrow{AD} - \overrightarrow{CG} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \quad \overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix}$$

Exercise 11H

$$1 \quad \vec{PQ} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x = 1 + \lambda, y = 3 + 2\lambda$$

$$x - 1 = \frac{y - 3}{2}$$

$$2x - 2 = y - 3$$

$$y = 2x + 1$$

$$2 \quad r = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$x = 1 + 2\lambda, y = -1 - \lambda, z = 1 + 3\lambda$$

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-1}{3}$$

$$3 \quad \frac{x+1}{3} = \frac{2y}{3} = z - 1 \Rightarrow \frac{x+1}{3} = \frac{y}{\frac{3}{2}} = \frac{z-1}{1}$$

$$(-1, 0, 1) \quad \begin{pmatrix} 3 \\ \frac{3}{2} \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

$$4 \quad a \quad \text{eg use } \lambda = 0, 1, -1$$

$$(1, 1, -1) \quad (0, 1, 2) \quad (2, 1, -4)$$

(other solutions are possible)

$$b \quad x = 1 - \lambda \quad y = 1 \quad z = -1 + 3\lambda$$

At P, $y = 3 \neq 1 \therefore$ P does not lie on L (QED)

$$c \quad r = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$5 \quad a \quad \text{eg use } k = 0, 1 \quad (1, 0, 2) \quad (2, -1, 2)$$

$$b \quad \text{direction of line} = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \pm \frac{4}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \pm (2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j}) = \pm \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix}$$

Exercise 11I

$$1 \quad \mathbf{u} \cdot \mathbf{v} = 1.5 \times 4 \times \cos 30^\circ = 3\sqrt{3}$$

$$2 \quad \mathbf{u} \cdot (-\mathbf{v}) = |\mathbf{u}| |\mathbf{v}| \cos(\pi - \theta)$$

$$= |\mathbf{u}| |\mathbf{v}| \cos(\pi - \theta)$$

$$= -|\mathbf{u}| |\mathbf{v}| \cos \theta$$

$$= -(\mathbf{u} \cdot \mathbf{v})$$

$$(-\mathbf{u}) \cdot \mathbf{v} = |-\mathbf{u}| |\mathbf{v}| \cos(\pi - \theta)$$

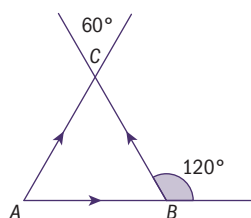
$$= |\mathbf{u}| |\mathbf{v}| \cos(\pi - \theta)$$

$$= -|\mathbf{u}| |\mathbf{v}| \cos \theta$$

$$= -(\mathbf{u} \cdot \mathbf{v})$$

$$\therefore \mathbf{u} \cdot (-\mathbf{v}) = -(\mathbf{u} \cdot \mathbf{v}) = (-\mathbf{u}) \cdot \mathbf{v} \quad (\text{QED})$$

$$3 \quad \text{Let the length of the sides be } x$$



$$\vec{AB} \cdot \vec{BC} = x^2 \cos 120^\circ = -\frac{1}{2}x^2$$

$$\vec{BC} \cdot \vec{AC} = x^2 \cos 60^\circ = \frac{1}{2}x^2$$

$$\therefore \vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{AC} = -\frac{1}{2}x^2 + \frac{1}{2}x^2 = 0$$

$$4 \quad a \quad \vec{AB} \cdot \vec{AC} = xy \cos \alpha = x^2$$

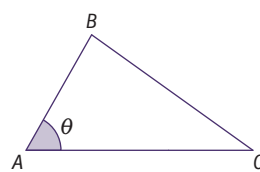
$$b \quad \vec{CA} \cdot \vec{CB}$$

$$= y\sqrt{y^2 - x^2} \cos\left(\frac{\pi}{2} - \alpha\right) = y\sqrt{y^2 - x^2} \sin \alpha = y^2 \sin^2 \alpha$$

$$c \quad \vec{AC} \cdot \vec{CB}$$

$$= y\sqrt{y^2 - x^2} \cos\left(\frac{\pi}{2} + \alpha\right) = -y\sqrt{y^2 - x^2} \sin \alpha = -y^2 \sin^2 \alpha$$

5



$$\text{Area} = 4$$

$$\therefore \frac{1}{2} \times 2 \times 5 \sin \theta = 4$$

$$\therefore \sin \theta = \frac{4}{5}$$

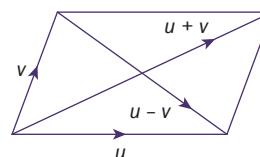
$$\therefore \cos \theta = \pm \frac{3}{5}$$

$$\vec{AB} \cdot \vec{AC} = 2 \times 5 \cos \theta = \pm 6$$

6 Angle between $\mathbf{u}$ and $\mathbf{u} = 0$

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}| |\mathbf{u}| \cos 0 = |\mathbf{u}|^2 \quad (\text{QED})$$

7



if $|\mathbf{u} + \mathbf{v}| = |\mathbf{u} - \mathbf{v}|$
the parallelogram is a
rectangle and the angle
between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{2}$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \frac{\pi}{2} = 0 \quad (\text{QED})$$

$$8 \quad \text{diagonal AG} = \sqrt{7^2 + 2^2 + 3^2} = \sqrt{62}$$

$$a \quad OB = OC = \frac{\sqrt{62}}{2} \cos \hat{BOC} = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 2^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{27}{31}$$

$$\vec{OB} \cdot \vec{OC} = \left(\frac{\sqrt{62}}{2}\right) \left(\frac{\sqrt{62}}{2}\right) \left(\frac{27}{31}\right) = \frac{27}{2}$$

$$b \quad OA = OE = \frac{\sqrt{62}}{2} \cos \hat{AOE} = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 3^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{22}{31}$$

$$\vec{OA} \cdot \vec{OE} = \left(\frac{\sqrt{62}}{2}\right) \left(\frac{\sqrt{62}}{2}\right) \left(\frac{22}{31}\right) = 11$$

Exercise 11J

$$1 \quad a \quad \mathbf{u} \cdot \mathbf{v} = -12 + 24 = 12$$

$$b \quad \mathbf{u} \cdot \mathbf{v} = -1 - 3 + 10 = 6$$

$$2 \quad A(-1, 3, 2) \quad B(-1, 1, 2) \quad C(1, -1, 1)$$

$$\vec{AB} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \quad \vec{BC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$$

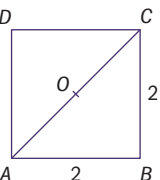
$$\vec{AB} \cdot \vec{BC} = 0 + 4 - 4 = 0$$

$$\vec{AC} \cdot \vec{BC} = 4 + 8 - 3 = 9$$

3 a $0(0, 0, 0)$ $A(0, 0, 1)$ $B(1, 0, 1)$ $C(1, 0, 0)$
 $D(0, 1, 0)$ $E(0, 1, 1)$ $F(1, 1, 1)$ $G(1, 1, 0)$

b $\vec{OF} \cdot \vec{OG} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2$

$\vec{AF} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{BG} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\vec{AF} \cdot \vec{BG} = 1$

4 a  $AC = 2\sqrt{2}$
 $\therefore OA = OB = OC = OD = \sqrt{2}$

Volume $= \frac{1}{3} \times 4 \times OE = \frac{8}{3} \therefore OE = 2$
 $A(0, \sqrt{2}, 0)$ $B(\sqrt{2}, 0, 0)$ $C(0, \sqrt{2}, 0)$
 $D(-\sqrt{2}, 0, 0)$ $E(0, 0, 2)$

b $\vec{EA} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ -2 \end{pmatrix}$ $|\vec{EA}| = \sqrt{6}$

$\vec{EB} = \begin{pmatrix} \sqrt{2} \\ 0 \\ -2 \end{pmatrix}$ $\vec{EA} \cdot \vec{EB} = 4$

c $\cos \hat{AEB} = \frac{6+6-4}{2\sqrt{6}\sqrt{6}} = \frac{8}{12}$
 $\hat{AEB} = 48.2^\circ$

Exercise 11K

1 $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
 $\mathbf{u} \cdot \mathbf{v} = 2 - 6 = -4$ $|\mathbf{u}| = \sqrt{13}$ $|\mathbf{v}| = \sqrt{5}$
 $\cos \theta = \frac{-4}{\sqrt{13}\sqrt{5}} \therefore \theta = 120^\circ$

2 $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
 $\mathbf{u} \cdot \mathbf{v} = 2 + 2 + 1 = 5$ $|\mathbf{u}| = \sqrt{6}$ $|\mathbf{v}| = \sqrt{6}$
 $\cos \theta = \frac{5}{6} \therefore \theta = 33.6^\circ$

3 $A(-1, 1, 1)$ $B(1, -1, 2)$ $C(2, 3, -1)$

a $\vec{AB} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$\vec{AB} \cdot \vec{AC} = 6 - 4 - 2 = 0$
 $\therefore \cos \theta = 0 \therefore \theta = 90^\circ$

b $\vec{BC} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ $\vec{BC} \cdot \vec{AC} = 3 + 8 + 6 = 17$

$|\vec{BC}| = \sqrt{26}$ $|\vec{AC}| = \sqrt{17}$

$\cos \theta = \frac{17}{\sqrt{26}\sqrt{17}} \therefore \theta = 36.0^\circ$

4 $\mathbf{u} \cdot \mathbf{v} > 0 \therefore a(a-2) + 3(a-4) > 0$
 $a^2 + a - 12 > 0$

When $(a-3)(a+4) > 0$

$a = 3$ or $-4 \therefore a < -4$ or $a > 3$

5 $\mathbf{u} = \sin 3\alpha \mathbf{i} - \cos 3\alpha \mathbf{j} + 2\mathbf{k}$

$\mathbf{v} = \cos \alpha \mathbf{i} - \sin \alpha \mathbf{j} - 2\mathbf{k}$

a $\mathbf{u} \cdot \mathbf{v} = \sin 3\alpha \cos \alpha + \cos 3\alpha \sin \alpha - 4$
 $= \sin 4\alpha - 4$

$|\mathbf{u}|^2 = \sin^2 3\alpha + \cos^2 3\alpha + 4 = 5 \therefore |\mathbf{u}| = \sqrt{5}$

$|\mathbf{v}|^2 = \cos^2 \alpha + \sin^2 \alpha + 4 = 5 \therefore |\mathbf{v}| = \sqrt{5}$

$\therefore \cos \theta = \frac{\sin 4\alpha - 4}{5}$

b $\cos 150^\circ = \frac{\sin 4\alpha - 4}{5}$

$-\frac{\sqrt{3}}{2} = \frac{\sin 4\alpha - 4}{5}$

$\sin 4\alpha = -0.3301$

$4\alpha = 3.478, 5.947, 9.761, 12.230, 16.044,$
 $18.513, 22.328, 27.796$

$\alpha = 0.870, 1.49, 2.44, 3.06, 4.01, 4.63, 5.58,$
 6.20

c $\sin 4\alpha < 4 \therefore \sin 4\alpha - 4 < 0$

$\therefore \cos \theta < 0 \therefore \theta$ is obtuse (QED)

6 Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$ $\mathbf{c} + \mathbf{d} = \begin{pmatrix} c_1 + d_1 \\ c_2 + d_2 \\ c_3 + d_3 \end{pmatrix}$

$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = (a_1 + b_1)(c_1 + d_1) + (a_2 + b_2)(c_2 + d_2) + (a_3 + b_3)(c_3 + d_3)$

$= a_1 c_1 + a_1 d_1 + b_1 c_1 + b_1 d_1 + a_2 c_2 + a_2 d_2 + b_2 c_2$
 $+ b_2 d_2 + a_3 c_3 + a_3 d_3 + b_3 c_3 + b_3 d_3$

$= (a_1 c_1 + a_2 c_2 + a_3 c_3) + (a_1 d_1 + a_2 d_2 + a_3 d_3)$
 $+ (b_1 c_1 + b_2 c_2 + b_3 c_3) + (b_1 d_1 + b_2 d_2 + b_3 d_3)$

$= \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$ (QED)

Exercise 11L

1 $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\mathbf{w} = \mathbf{i} - \mathbf{k}$

a $\mathbf{u} \times \mathbf{v} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

b $\mathbf{v} \times \mathbf{w} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

c $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -4\mathbf{j} + 4\mathbf{k} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix}$

$$\mathbf{d} \quad (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$

$$\mathbf{e} \quad \mathbf{u} \times \mathbf{w} = \mathbf{i} + \mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{f} \quad (\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{g} \quad (\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{w}) = (2\mathbf{i} - 3\mathbf{j}) \times (-\mathbf{j}) = -2\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

$$2 \quad \mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k} \quad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \mathbf{w} = \mathbf{i} - \mathbf{k}$$

From qn. 1, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -4\mathbf{j} + 4\mathbf{k}$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\therefore \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

$\therefore$ not associative (QED)

$$3 \quad \mathbf{w} \cdot \mathbf{u} = 0 \text{ and } \mathbf{w} \cdot \mathbf{v} = 0$$

$$\therefore w_1 u_1 + w_2 u_2 + w_3 u_3 = 0 \text{ and}$$

$$w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

$$\begin{aligned} \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= [w_2 (u_1 v_2 - u_2 v_1) - w_3 (u_3 v_1 - u_1 v_3)] \mathbf{i} \\ &\quad + [w_3 (u_2 v_3 - u_3 v_2) - w_1 (u_1 v_2 - u_2 v_1)] \mathbf{j} \\ &\quad + [w_1 (u_3 v_1 - u_1 v_3) - w_2 (u_2 v_3 - u_3 v_2)] \mathbf{k} \\ &= [u_1 (w_2 v_2 + w_3 v_3) - v_1 (w_2 u_2 + w_3 u_3)] \mathbf{i} \\ &\quad + [u_2 (w_3 v_3 + w_1 v_1) - v_2 (w_3 u_3 + w_1 u_1)] \mathbf{j} \\ &\quad + [u_3 (w_1 v_1 + w_2 v_2) - v_3 (w_1 u_1 + w_2 u_2)] \mathbf{k} \\ &= [u_1 (-w_1 v_1) - v_1 (-w_1 u_1)] \mathbf{i} \\ &\quad + [u_2 (-w_2 v_2) - v_2 (-w_2 u_2)] \mathbf{j} \\ &\quad + [u_3 (-w_3 v_3) - v_3 (-w_3 u_3)] \mathbf{k} = 0 \end{aligned}$$

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0 \therefore \mathbf{w} \text{ and } \mathbf{u} \times \mathbf{v} \text{ are collinear}$$

$$4 \quad A(-1, 3, 4) \quad B(5, 7, 5) \quad C(3, 9, 6)$$

$$\mathbf{a} \quad \overrightarrow{AB} = \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix}$$

$$\therefore D(-3, 5, 5)$$

$$\mathbf{b} \quad \overrightarrow{AB} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \quad \overrightarrow{BC} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} 2 \\ -8 \\ 20 \end{pmatrix}$$

$$\begin{aligned} \mathbf{c} \quad \text{area} &= |\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{4 + 64 + 400} \\ &= \sqrt{468} = 2\sqrt{117} \\ &= 6\sqrt{13} \end{aligned}$$

$$5 \quad \mathbf{a} \quad A(1, 0, 2) \quad B(2, 3, 3) \quad C(-3, -1, 2) \quad E(2, 1, 4)$$

$$\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix} \quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} -4 \\ -4 \\ 1 \end{pmatrix}$$

$$D(-4, -4, 1)$$

$$\mathbf{b} \quad \text{Volume} = (\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 11 \end{pmatrix}$$

$$\text{Volume} = \begin{pmatrix} 1 \\ -4 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 - 4 + 22$$

$$= 19 \text{ cu. units}$$

$$\begin{aligned} 6 \quad \mathbf{a} \quad \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} \\ &\quad + (u_1 v_2 - u_2 v_1) \mathbf{k} \\ &= -(v_2 u_3 - v_3 u_2) \mathbf{i} - (v_3 u_1 - v_1 u_3) \mathbf{j} \\ &\quad - (v_1 u_2 - v_2 u_1) \mathbf{k} \end{aligned}$$

$$\mathbf{v} \times \mathbf{u} = (v_2 u_3 - v_3 u_2) \mathbf{i} + (v_3 u_1 - v_1 u_3) \mathbf{j} + (v_1 u_2 - v_2 u_1) \mathbf{k}$$

$$\therefore \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \quad (\text{QED})$$

$$\mathbf{b} \quad \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$$

$$\begin{aligned} &= [u_2 (v_3 + w_3) - u_3 (v_2 + w_2)] \mathbf{i} + [u_3 (v_1 + w_1) - u_1 (v_3 + w_3)] \mathbf{j} \\ &\quad + [u_1 (v_2 + w_2) - u_2 (v_1 + w_1)] \mathbf{k} \\ &= [(u_2 v_3 - u_3 v_2) + (u_2 w_3 - u_3 w_2)] \mathbf{i} \\ &\quad + [(u_3 v_1 - u_1 v_3) + (u_3 w_1 - u_1 w_3)] \mathbf{j} \\ &\quad + [(u_1 v_2 - u_2 v_1) + (u_1 w_2 - u_2 w_1)] \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} \\ &\quad + (u_1 v_2 - u_2 v_1) \mathbf{k} \\ &\quad + (u_2 w_3 - u_3 w_2) \mathbf{i} + (u_3 w_1 - u_1 w_3) \mathbf{j} \\ &\quad + (u_1 w_2 - u_2 w_1) \mathbf{k} \end{aligned}$$

$$\therefore \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \quad (\text{QED})$$

$$\mathbf{c} \quad \mathbf{u} \times \mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_1 u_2 v_3 + u_1 u_3 v_2 - u_2 u_3 v_1 = 0$$

$$\therefore \mathbf{u} \times \mathbf{v} \cdot \mathbf{u} = 0 \quad (\text{QED})$$

$$\mathbf{d} \quad (\lambda \mathbf{u}) \times \mathbf{v} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda u_2 v_3 - \lambda u_3 v_2 \\ \lambda u_3 v_1 - \lambda u_1 v_3 \\ \lambda u_1 v_2 - \lambda u_2 v_1 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \lambda (\mathbf{u} \times \mathbf{v})$$

$$\therefore (\lambda \mathbf{u}) \times \mathbf{v} = \lambda (\mathbf{u} \times \mathbf{v}) \quad (\text{QED})$$

Exercise 11M

1 A(-3, 1, 1) B(1, 2, 0) C(1, 1, -2)

a $\vec{AB} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

$$r = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

b $x = -3 + 4\alpha + 4\beta$ (1)

$y = 1 + \alpha$ (2)

$z = 1 - \alpha - 3\beta$ (3)

c From (2) $\alpha = y - 1$

From (1) $4\beta = x + 3 - 4(y - 1)$

$$\beta = \frac{x - 4y + 7}{4}$$

sub in (3): $z = 1 - y + 1 - \frac{3}{4}(x - 4y + 7)$

$4z = 8 - 4y - 3x + 12y - 21$

$3x - 8y + 4z = -13$ (other forms are possible)

2 P(1, 0, 1) $\mathbf{n} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$-x + 3y - z = -2$

$x - 3y + z = 2$

eg (0, 0, 2) (0, $-\frac{2}{3}$, 0) (2, 0, 0)

3 a (-3, 4, 0) $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$x - 2y - z = -11$

b P(-1, 1, 1) $x - 1 = \frac{y-1}{2} = z$

line passes through A(1, 1, 0) and has direction $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\vec{AP} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$-2x + 3y - 4z = 1$

$2x - 3y + 4z = -1$

c $1 - x = y - 1 = 2z$ and $x = 2 - t$

$$\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{\frac{1}{2}} \quad y = 1 + 2t$$

$z = t$

(2, 1, 0) lies in the plane

$\begin{pmatrix} -1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ are vectors in the plane

$$\mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$y - 2z = 1$

4 a $y = 0$

b $z = 0$

c $x = 0$

5 a $\frac{1}{3} \times 4 \times h = 6 \therefore h = \frac{9}{2}$

0(0, 0, 0) A(2, 0, 0) B(2, 2, 0) C(0, 2, 0)

V(0, 0, $\frac{9}{2}$)

b $\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $\vec{AV} = \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$$

c $\vec{CB} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\vec{CV} = \begin{pmatrix} 0 \\ -2 \\ \frac{9}{2} \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -9 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -9 \\ -4 \end{pmatrix}$$

$-9y - 4z = -18$

$9y + 4z = 18$

d $\vec{VB} = \begin{pmatrix} 2 \\ 2 \\ -\frac{9}{2} \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ -9 \end{pmatrix}$

$$\begin{aligned} \mathbf{e} \quad \mathbf{r} &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 9 \end{pmatrix} \\ x &= 2 - 4\lambda, y = 0, z = 9\lambda \\ \lambda &= \frac{x-2}{-4} \quad \lambda = \frac{z}{9} \\ \frac{x-2}{-4} &= \frac{z}{9} \Rightarrow 9x - 36 = -4z \\ z &= \frac{36-9x}{4}, y = 0 \end{aligned}$$

Exercise 11N

$$1 \quad \mathbf{r} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad y = \sqrt{3}x - 3$$

$$\text{Method 1} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$\cos \theta = \frac{|1-\sqrt{3}|}{\sqrt{2}\sqrt{4}} \therefore \theta = 75^\circ$$

$$\text{Method 2} \quad m_1 = -1 \quad m_2 = \sqrt{3}$$

$$\tan \mu = -1 \quad \tan \beta = \sqrt{3}$$

$$\therefore \theta = -45^\circ \therefore \beta = 60^\circ$$

$$\theta = 1\beta - \theta_1 = 105^\circ$$

$$\therefore \text{acute angle} = 75^\circ$$

$$2 \quad \frac{x-1}{2} = \frac{y}{3} = \frac{z}{1} \quad 2x = y = 3z$$

$$\frac{x}{\frac{1}{2}} = \frac{y}{1} = \frac{z}{\frac{1}{3}}$$

$$\frac{x}{\frac{1}{2}} = \frac{y}{1} = \frac{z}{\frac{1}{3}}$$

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 6 + 18 + 2 = 26$$

$$|\mathbf{u}| = \sqrt{14} \quad |\mathbf{v}| = 7$$

$$\cos \theta = \frac{26}{7\sqrt{14}} \therefore \theta = 6.93^\circ$$

$$3 \quad \mathbf{r} = (2 - \theta)\mathbf{i} + (-2\theta)\mathbf{j} + (1 - \theta)\mathbf{k}$$

$$\mathbf{r} = (2 + \beta)\mathbf{j} + (3 + \beta)\mathbf{k}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 1$$

$$\cos \theta = \frac{1}{\sqrt{6}\sqrt{2}} \therefore \theta = 73.2^\circ$$

$$4 \quad \mathbf{a} \quad \mathbf{u} = \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix} \quad \mathbf{u} \cdot \mathbf{v} = 1 + m_1 m_2$$

$$|\mathbf{u}| = \sqrt{1+m_1^2} \quad |\mathbf{v}| = \sqrt{1+m_2^2}$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}| |\mathbf{v}|}$$

$$\therefore \cos \theta = \frac{|1+m_1 m_2|}{\sqrt{1+m_1^2} \sqrt{1+m_2^2}}$$

$$\mathbf{b} \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\mathbf{c} \quad \sec^2(\alpha - \beta) = 1 + \tan^2(\alpha - \beta)$$

$$= 1 + \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

$$= \frac{(1 + m_1 m_2)^2 + (m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

$$= \frac{1 + 2m_1 m_2 + m_1^2 m_2^2 + m_1^2 - 2m_1 m_2 + m_2^2}{(1 + m_1 m_2)^2}$$

$$= \frac{1 + m_1^2 + m_2^2 + m_1^2 m_2^2}{(1 + m_1 m_2)^2}$$

$$= \frac{(1 + m_1^2)(1 + m_2^2)}{(1 + m_1 m_2)^2}$$

$$\therefore \cos(\alpha - \beta) = \frac{|1 + m_1 m_2|}{\sqrt{1 + m_1^2} \sqrt{1 + m_2^2}} (> 0 \text{ since } \alpha > \beta)$$

$$\mathbf{d} \quad \cos \theta = \cos(\alpha - \beta) \text{ since } \theta = \alpha - \beta$$

Exercise 11O

$$1 \quad \mathbf{r} = (1 - 2\lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + (-2 + \lambda)\mathbf{k} \quad \mathbf{u} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$2x - y + z = 5 \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\sin \theta = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{u}| |\mathbf{n}|} = \frac{|-2|}{\sqrt{6}\sqrt{6}} \quad \theta = 195^\circ$$

$$2 \quad \frac{x-1}{3} = 2y = 3 - 2z \Rightarrow \frac{x-1}{3} = \frac{y}{\frac{1}{2}} = \frac{z-\frac{3}{2}}{\frac{-1}{2}}$$

$$\Rightarrow \frac{x-1}{6} = \frac{y}{1} = \frac{z-\frac{3}{2}}{-1} \Rightarrow \mathbf{u} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = (2 - 2\theta - 3\beta)\mathbf{i} + (1 - \theta + \beta)\mathbf{j} + (-2\theta + \beta)\mathbf{k}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$\mathbf{u} \cdot \mathbf{n} = 6 + 8 + 5 = 19$$

$$\sin \theta = \frac{19}{\sqrt{38}\sqrt{90}} \therefore \theta = 19.0^\circ$$

$$3 \quad x - y + 3z = 1 \quad \mathbf{m} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$\mathbf{r} = (4 - 2\theta + 2\beta)\mathbf{i} + (1 - 3\beta)\mathbf{j} + (2 - \theta - \beta)\mathbf{k}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \quad \mathbf{m} \cdot \mathbf{n} = -3 + 4 + 18 = 19$$

$$\cos \theta = \frac{19}{\sqrt{11}\sqrt{61}} \quad \theta = 42.8^\circ$$

4 A(1, 0, 1) B(-1, 1, 0) C(2, 3, -1) D(-1, -1, -1)

a $\vec{AB} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$

$x - 5y - 7z = -6$

b $\vec{AD} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$x = 1 + 2\lambda$ $y = \lambda$ $z = 1 + 2\lambda$
 $\frac{x-1}{2} = y = \frac{z-1}{2}$

c $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$ $\mathbf{u} \cdot \mathbf{n} = 2 - 5 - 14 = -17$

$\sin \theta = \frac{17}{3\sqrt{75}}$ $\theta = 40.9^\circ$

5 $\frac{x}{2} = ky = k - z$ $(2k - 1)x - ky + z = 5 + k$

$\frac{x}{-2k} = \frac{y}{-1} = \frac{z-k}{k}$ $\mathbf{u} = \begin{pmatrix} -2k \\ -1 \\ k \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 2k-1 \\ -k \\ 1 \end{pmatrix}$

If the line and plane are parallel, $\mathbf{u} \cdot \mathbf{n} = 0$

so $-2k(2k - 1) + k + k = 0$

$\Rightarrow -4k^2 + 4k = 0$

$\Rightarrow -4k(k - 1) = 0 \Rightarrow k = 0$ or 1

Exercise 11P

1 $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$P(1 + \lambda, -2\lambda, 1 + \lambda)$ $x + y + 2z = 4$

$1 + \lambda - 2\lambda + 2(1 + \lambda) = 4$

$\lambda = 1$ $P(2, -2, 2)$

2 $\frac{x-1}{5} = \frac{y}{2} = \frac{z}{3} = \lambda$ $x = 1 + 5\lambda$, $y = 2\lambda$, $z = 3\lambda$

$-x - y + 3z = 5$

$-1 - 5\lambda - 2\lambda + 9\lambda = 5$

$\therefore 2\lambda = 6 \therefore \lambda = 3$ (16, 6, 9)

3 $x = 3k$, $y = 2 - 2k$, $z = 1 - k$

$\mathbf{r} = (4 - 2\theta + \beta)\mathbf{i} + (1 - \beta)\mathbf{j} + (2 - \theta - 2\beta)\mathbf{k}$

a $x = 4 - 2\theta + \beta$

$y = 1 - \beta$

$z = 2 - \theta - 2\beta$

b $3k = 4 - 2\theta + \beta \Rightarrow 3k + 2\theta - \beta = 4$

$2 - 2k = 1 - \beta \Rightarrow -2k + \beta = -1$

$1 - k = 2 - \theta - 2\beta \Rightarrow k + \theta + 2\beta = 1$

$k = 0.6$, $\theta = 1.2$, $\beta = 0.2$ (1.8, 0.8, 0.4)

4 $\mathbf{r} = (1 + \lambda)\mathbf{i} + (1 + 2\lambda)\mathbf{j} + (1 + \lambda)\mathbf{k}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$3x - y - z = 2$ $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$\mathbf{u} \cdot \mathbf{n} = 3 - 2 - 1 = 0 \therefore \mathbf{u}$ and $\mathbf{n}$ are perpendicular

$\therefore$ the line parallel to the plane (QED)

(0, 0, -2) lies in the plane

$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ lies in the plane

$\mathbf{r} = \lambda\mathbf{i} + 2\lambda\mathbf{j} + (-2 + \lambda)\mathbf{k}$ (other answers are possible)

5 $P(1, 2, 3)$ $2x + y - 5z = 1$ $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

line through p perpendicular to the plane:

$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

point of intersection, $I(1 + 2\lambda, 2 + \lambda, 3 - 5\lambda)$

$2(1 + 2\lambda) + 2 + \lambda - 5(3 - 5\lambda) = 1$

$2 + 4\lambda + 2 + \lambda - 15 + 25\lambda = 1$

$30\lambda = 12 \Rightarrow \lambda = 0.4$

$I(1.8, 2.4, 1)$

$PI = \sqrt{(1.8-1)^2 + (2.4-2)^2 + (1-3)^2}$
 $= \sqrt{4.8} = \frac{1}{10}\sqrt{480}$

distance $= \frac{2}{5}\sqrt{30}$ or 2.19

Exercise 11Q

1 a $\frac{x}{2} = y - 1 = z$ (1) $x = \frac{y+4}{3} = 3 - z$ (2)

$x = 2y - 2$ $3x = y + 4$

$6y - 6 = y + 4$

$5y = 10 \Rightarrow y = 2$, $x = 2$

sub in (1) sub in (2)

$\frac{2}{2} = 2 - 1 = z$ $2 = \frac{2+4}{3} = 3 - z$

$z = 1$ $z = 1$

$\therefore$ intersect at (2, 2, 1)

b $\mathbf{r} = (5 + 2\lambda)\mathbf{i} + (4 + \lambda)\mathbf{j} + (5 - 3\lambda)\mathbf{k}$, $x = y = z + 1$

$P(5 + 2\lambda, 4 + \lambda, 5 - 3\lambda)$

$5 + 2\lambda = 4 + \lambda$ and $4 + \lambda = 5 - 3\lambda + 1$

$\lambda = -1$ $4\lambda = 2$
 $\lambda = \frac{1}{2}$

Equations are inconsistent $\therefore$ no point of intersection

$$\begin{aligned}
 2 \quad & x - 3y + 8 = 2 \quad (1) \\
 & -x + y - 2z = 1 \quad (2) \\
 & (1) + (2) - 2y - z = 3 \Rightarrow z = -2y - 3 \\
 & 3(2) + (1) - 2x - 5z = 5 \Rightarrow z = \frac{-2x-5}{5} \\
 & \frac{-2x-5}{5} = -2y - 3 = z \\
 & \frac{2x+5}{5} = 2y + 3 = \frac{z}{-1}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad & \mathbf{a} \quad 3x - y + z = 3 \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\
 & x = 3 - (2 - k) \lambda \quad y = (2k - 1) + \lambda
 \end{aligned}$$

$$z = -1 + k\lambda \quad \mathbf{u} = \begin{pmatrix} k-2 \\ 1 \\ k \end{pmatrix}$$

$\mathbf{n}$ and $\mathbf{u}$ are collinear $\therefore k = -1$

$$\begin{aligned}
 \mathbf{b} \quad & 3x - y + z = 3 \quad x = 3 - 3\lambda, y = -3 + \lambda, \\
 & z = -1 - \lambda \\
 & 3(3 - 3\lambda) - (-3 + \lambda) + (-1 - \lambda) = 3 \\
 & 9 - 9\lambda + 3 - \lambda - 1 - \lambda = 3 \\
 & -11\lambda = -8 \\
 & \lambda = \frac{8}{11} \left(\frac{9}{11}, \frac{-25}{11}, \frac{-19}{11} \right)
 \end{aligned}$$

$$\begin{aligned}
 4 \quad & \mathbf{a} \quad L_1: y = 2x + 2, z = 3 - x \\
 & \frac{x}{1} = \frac{y-2}{2} = \frac{z-3}{-1} \text{ direction, } \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
 & L_2: \frac{x-1}{3} = \frac{y-1}{6} = \frac{1-z}{3} \Rightarrow \frac{x-1}{3} = \frac{y-1}{6} = \frac{z-1}{-3} \\
 & \text{direction, } \mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}
 \end{aligned}$$

$\mathbf{v} = 3\mathbf{u} \therefore L_1$ and L_2 are parallel (QED)

$\mathbf{b} \quad A(0, 2, 3)$ and $B(1, 1, 1)$ are points in the plane

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$5x - y + 3z = 7$$

$$5 \quad x + y + 3z = 1 \quad x = 4 - y \text{ and } z = -1$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad \frac{x}{1} = \frac{y-4}{-1} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{u} \cdot \mathbf{n} = 1 - 1 + 0 = 0$$

$\therefore \mathbf{u}$ and $\mathbf{n}$ are perpendicular $\therefore$ the line is parallel to the plane.

$(0, 4, -1)$ lies on the line

$$x + y + 3z = 0 + 4 + 3(-1) = 1$$

$\therefore (0, 4, -1)$ also lies in the plane

$\therefore$ the plane contains the line (QED)

Exercise 11R

$$\begin{aligned}
 1 \quad & 5x + y + 2z = 3 \quad (1) \quad (1) - (3) \quad x - y = -2 \quad (4) \\
 & x + y + z = 3 \quad (2) \quad (1) - 2(2) \quad 3x - y = -3 \quad (5) \\
 & 4x + 2y + 2z = 5 \quad (3) \quad (5) - (4) \quad 2x = -1 \\
 & \therefore x = \frac{-1}{2} \quad y = \frac{3}{2}
 \end{aligned}$$

$$\text{From (2)} \quad z = 3 - x - y = 2 \left(\frac{-1}{2}, \frac{3}{2}, 2 \right)$$

$$\begin{aligned}
 3 \quad & x + y + z = 1 \quad (1) \quad (1) - (2) \quad 2y = -2 \\
 & x - y + z = 3 \quad (2) \quad y = -1 \\
 & 3x + y + 3z = 1 \quad (3)
 \end{aligned}$$

$$\text{sub in (1) and (3)} \quad x + z = 2 \quad (4)$$

$$3x + 3z = 2 \quad (5)$$

(4) and (5) are inconsistent $\therefore$ no common point (QED)

$$\begin{aligned}
 4 \quad & \mathbf{a} \quad x + y + z = 0 \quad (1) \quad (2) - (1) \quad ax - x = 0 \\
 & ax + y + z = 0 \quad (2) \quad x(a - 1) = 0 \\
 & x + by + cz = z = 0 \quad (3) \quad x = 0 \text{ or } a = 1 \\
 & \text{if } x = 0, y + z = 0 \text{ either } y = z = 0 \\
 & by + cz = 0 \text{ or } b = c
 \end{aligned}$$

If $x = 0$, either point of intersection $(0, 0, 0)$ or intersect in a straight line.

if $a = 1$ equations (1) and (2) are the same

$$x + y + z = 0$$

$$x + by + cz = 0$$

$x = y = z = 0$ or $b = c = 1$ all 3 planes the same or b and c not both = 1 and intersect in a straight line.

$\therefore$ if $a = 1$, either a point of intersection $(0, 0, 0)$ or intersect in a plane or intersect in a line.

$\therefore$ the planes always have at least one point in common.

$$\begin{aligned}
 \mathbf{b} \quad & \text{For a straight line, } b = c \text{ but } a, b, c \text{ not all} \\
 & \text{equal to 1 or } a = 1 \text{ and } b \text{ and } c \text{ both equal to 1} \\
 & x + 2y - 2z = 5 \quad \pi_1 \\
 & 3x - 6y + 3z = 2 \quad \pi_2 \\
 & x - 2y + z = 7 \quad \pi_3
 \end{aligned}$$

π_2 and π_3 are parallel but not coincident, π_1 intersects each of the other 2 planes in a straight line but the 3 planes have no point in common.

- 6 a $x + y + z = 2$ (1) (1) + (2) $3x + 2z = 1$ (4)
 $2x - y + z = -1$ (2) (1) + (3) $4x - 2z = 6$ (5)
 $3x - y - 3z = 4$ (3) (4) + (5) $7x = 7$
 $x = 1, z = -1, y = 2$
 $(1, 2, -1)$
- b $x + y + z = 2$ (1) (1) + (2) $3x + 2z = 1$
 $2x - y + z = -1$ (2) (1) + (3) $4x + (k + 1)z = 6$
 $3x - y + kz = 4$ (3)
 For no common point, $\frac{4}{3} = \frac{k+1}{2} (\neq 6)$
 $8 = 3k + 3 \therefore k = \frac{5}{3}$

7 Verify by diagrams

Exercise 11S

- 1 A $x = 3 - t$ B $x = 4 - 3t$
 $y = 2t - 4$ $y = 3 - 2t$
- a A(3, -4) B(4, 3)
- b $V_A = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $V_B = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$
- c $\cos \theta = \frac{V_A \cdot V_B}{|V_A| |V_B|} = \frac{3-4}{\sqrt{5}\sqrt{13}} = \frac{-1}{\sqrt{65}} \quad \theta = 97.1^\circ$
- d $\overrightarrow{AB} = \begin{pmatrix} 4-3t \\ 3-2t \end{pmatrix} - \begin{pmatrix} 3-t \\ 2t-4 \end{pmatrix} = \begin{pmatrix} 1-2t \\ 7-4t \end{pmatrix}$
 $AB = \sqrt{(1-2t)^2 + (7-4t)^2}$
 This is a minimum when $t = 1.5$ hours
- 2 a $\overrightarrow{OP} = (5 + 10t)\mathbf{i} + (20 - 20t)\mathbf{j} + (30t - 10)\mathbf{k}$,
 $t > 0 \quad t = 0, P(5, 20, -10)$
- b Cartesian equations:
 $\frac{x-5}{10} = \frac{y-20}{-20} = \frac{z+10}{30} (\times 10)$
 $\frac{x-5}{1} = \frac{y-20}{-2} = \frac{z+10}{3}$
- c i $x + y + z = 55$
 $5 + 10t + 20 - 20t + 30t - 10 = 55$
 $20t = 40$
 $t = 2$
- ii (25, -20, 50)
- iii $P_0 P_2 = \begin{pmatrix} 20 \\ -40 \\ 60 \end{pmatrix}$ distance
 $= \sqrt{20^2 + (-40)^2 + 60^2} = 20\sqrt{14}$
- d $\overrightarrow{OQ} = \begin{pmatrix} 2t^2 \\ 1-2t \\ 1+t^2 \end{pmatrix} t > 0$
- i $\overrightarrow{PQ} = \begin{pmatrix} 2t^2 - 5 - 10t \\ 1 - 2t - 20 + 20t \\ 1 + t^2 - 30t + 10 \end{pmatrix} = \begin{pmatrix} 2t^2 - 10t - 5 \\ 18t - 19 \\ t^2 - 30t + 11 \end{pmatrix}$

$$PQ = \sqrt{(2t^2 - 10t - 5)^2 + (18t - 19)^2 + (t^2 - 30t + 11)^2}$$

This is a minimum when $t = 0.49598817$
 $= 0.496$ (3sf)

- ii P(9.96, 10.1, 4.88)
 Q(0.492, 0.00802, 1.25)

e i $a = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} b = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} c = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$

$$a - b = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} b - c = \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix}$$

$a - b \neq k(b - c) \therefore a - b$ and $b - c$ are non collinear (QED)

ii Q is not moving in a straight line

3 a B(1, 1, 0) C(0, 1, 0) $\therefore \overrightarrow{OP} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

C(0, 1, 0) G(0, 1, 1) $\therefore \overrightarrow{OQ} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$

D(0, 0, 1) G(0, 1, 1) $\therefore \overrightarrow{OR} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \overrightarrow{PR} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$$n = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

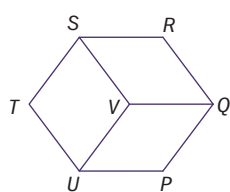
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x + y + z = \frac{3}{2} \text{ or } 2x + 2y + 2z = 3$$

$$\text{c } E(1, 0, 1) D(0, 0, 1) \therefore \vec{OS} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$A(1, 0, 0) E(1, 0, 1) \therefore \vec{OT} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$A(1, 0, 0) B(1, 1, 0) \therefore \vec{OU} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$



S, T, U lie in the plane PQR

$$\vec{PQ} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \vec{QR} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad \vec{RS} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \quad \vec{ST} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$\vec{TU} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \vec{UP} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

all sides are of length $\frac{1}{\sqrt{2}}$

$$\cos P = \cos Q = \cos R = \cos S = \cos T$$

$$= \cos U = \frac{-\frac{1}{4}}{\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = \frac{-1}{2}$$

$\therefore$ all angles are $120^\circ \therefore$ PQRSTU is a regular hexagon

$$\text{Area parallelogram PQVU} = |\vec{PQ} \times \vec{PU}|$$

$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \therefore \text{area PQVU} = \frac{\sqrt{3}}{4}$$

$$\therefore \text{area hexagon} = \frac{3\sqrt{3}}{4}$$

$$\text{d } \vec{OF} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = n \therefore \vec{OF} \text{ is perpendicular to the plane PQR (QED)}$$

e General point on OF is $(\lambda, \lambda, \lambda)$

$$2x + 2y + 2z = 3 \quad 2\lambda + 2\lambda + 2\lambda = 3$$

$$\therefore \lambda = \frac{3}{6} = \frac{1}{2} \quad I\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\text{f } \vec{IF} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \therefore \text{distance IF} = \frac{\sqrt{3}}{2}$$



Review exercise

$$1 \quad \mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad \mathbf{a} \quad 3\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} - \begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$$

$$\text{b } |\mathbf{u}| = 5 \quad |\mathbf{v}| = \sqrt{26} \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 9 \end{pmatrix} \therefore |\mathbf{u} + \mathbf{v}| = \sqrt{85}$$

$$2 \quad \mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \mathbf{a} = \mathbf{i} + \mathbf{j} \quad \mathbf{b} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{c} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\therefore \alpha + \beta + 2\gamma = 3 \quad (1) \quad (1) - (2) \quad \beta + 3\gamma = 4 \quad (4)$$

$$\alpha - \gamma = -1 \quad (2) \quad \beta - \gamma = 1 \quad (3)$$

$$\beta - \gamma = 1 \quad (3) \quad (4) - (3) \quad 4\gamma = 3$$

$$\gamma = \frac{3}{4}$$

$$\gamma = \frac{3}{4}, \beta = \frac{7}{4}, \alpha = -\frac{1}{4} \therefore \mathbf{u} = -\frac{1}{4}\mathbf{a} + \frac{7}{4}\mathbf{b} + \frac{3}{4}\mathbf{c}$$

$$3 \quad \mathbf{a} \quad \mathbf{a} = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} \quad |\mathbf{a}| = 5\sqrt{2} \therefore \text{unit vector} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$\text{b } \pm \frac{5}{5\sqrt{2}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = \pm \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$4 \quad \mathbf{u} = \cos \alpha \cos \beta \mathbf{i} + \sin^2 \alpha \cos^2 \beta \mathbf{j} + \sin^2 \beta \mathbf{k}$$

$$|\mathbf{u}|^2 = \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \cos^2 \beta + \sin^2 \beta$$

$$= \cos^2 \beta (\cos^2 \alpha + \sin^2 \alpha) + \sin^2 \beta$$

$$= \cos^2 \beta + \sin^2 \beta$$

$$= 1$$

$$\therefore |\mathbf{u}| = 1 \therefore \mathbf{u} \text{ is a unit vector}$$

$$5 \quad \mathbf{u} = \mathbf{i} + \tan \alpha \mathbf{j} \quad \mathbf{v} = \tan \beta \mathbf{i} + \mathbf{j}$$

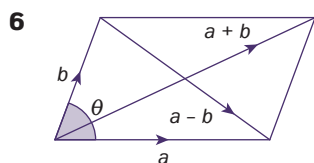
$$\cos r = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{\tan \beta + \tan \alpha}{\sqrt{1 + \tan^2 \alpha} \sqrt{1 + \tan^2 \beta}}$$

$$= \frac{\tan \beta + \tan \alpha}{\sec \alpha \sec \beta}$$

$$= (\tan \beta + \tan \alpha) \cos \alpha \cos \beta$$

$$= \sin \beta \cos \alpha + \sin \alpha \cos \beta$$

$$\begin{aligned}\cos \gamma &= \sin(\alpha + \beta) \\ &= \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) \\ \therefore \gamma &= \frac{\pi}{2} - (\alpha + \beta) \therefore \alpha + \beta + \gamma = \frac{\pi}{2}\end{aligned}$$



a $|a| = 1$ $|b| = 1$
 $|a - b|^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta$
 $|a - b|^2 = 2 - 2\cos\theta$
 $|a + b|^2 = 1^2 + 1^2 - 2(1)(1)\cos(\pi - \theta)$
 $= 2 + 2\cos(\pi - \theta)$

$$|a + b|^2 = 2 + 2\cos\theta$$

b $|a + b| = 2|a - b| \Rightarrow |a + b|^2 = 4|a - b|^2$
 $2 + 2\cos\theta = 4(2 - 2\cos\theta)$
 $2 + 2\cos\theta = 8 - 8\cos\theta$
 $10\cos\theta = 6 \therefore \cos\theta = \frac{3}{5} \quad (0 \leq \theta \leq \frac{\pi}{2})$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25} \therefore \sin\theta = \frac{4}{5}$$

7 a $r = (2i + 3j + 4k) + \alpha(2i - 3j + 2k) + \beta(j - 2k)$
 $x = 2 + 2\alpha, y = 3 - 3\alpha + \beta, z = 4 + 2\alpha - 2\beta$

b $r = (i - 2k) + \alpha(-2i + k) + \beta(-j)$
 $x = 1 - 2\alpha, y = -\beta, z = -2 + \alpha$

8 $5 + 3\lambda = -2 + 4\mu$ (1)
 $1 - 2\lambda = 2 + \mu$ (2)
 From (2) $\mu = -2\lambda - 1$
 Sub in (1) $5 + 3\lambda = -2 - 8\lambda - 4$
 $11\lambda = -11 \therefore \lambda = -1 \quad \alpha = 1 \quad P(2, 3)$



Review exercise

1 $u = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad v = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \cos\theta = \frac{u \cdot v}{|u||v|}$
 $\cos\theta = \frac{2+3}{\sqrt{11}\sqrt{5}} = \frac{5}{\sqrt{55}}$
 $\theta = 48^\circ$ (nearest degree)

2 a $\vec{AB} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$
 $r = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$
 $x = 1 - 2\alpha - \beta, y = 1, z = 2\alpha - \beta$
 $n = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$$

$$-4y = -4 \text{ or } y = 1$$

b $\vec{AB} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$$r = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$x = -1 - \alpha + \beta, y = 1 - 2\alpha, z = 1 + \alpha - 2\beta$$

$$n = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$4x - y + 2z = -3$$

3 a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\text{or } r \cdot (i + j - 2k) = -9$$

b $r \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad r(i - j) = 1$

4 a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

$$2x - 3y + 4z = 29$$

b $\vec{AB} = \begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$$

$$-2x - y + 4z = -12$$

$$2x + y - 4z = 12$$

c $\vec{AB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad n = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$2x - y = 4$$

$$\text{d } \vec{AB} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$5y + 2z = -11$$

$$\text{e } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x = 3$$

$$\text{f } \mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$x - y - z = 0$$

$$\text{g } x + y + 5z = 0 \quad (1)$$

$$2x + 3y + 12z = 0 \quad (2)$$

$$2 - 2(1)y + 2z = 0$$

$$\text{Let } z = \lambda, y = -2\lambda, x = -y - 5z = 2\lambda - 5\lambda$$

$$x = -3\lambda$$

$$\text{Line of intersection is } \mathbf{r} = \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{n} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$-x + 2y + z = 0$$

$$\text{or } x - 2y - z = 0$$

$$\text{5 } \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}| |\mathbf{v}|} = \frac{|2 - 1 - 2|}{\sqrt{6} \sqrt{6}} = \frac{1}{6}$$

$$\theta = 80.4^\circ$$

$$\text{6 } \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + \alpha(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$1 + \lambda = 1 + 2\alpha \quad (1)$$

$$1 + 2\lambda = 4 + \alpha \quad (2)$$

$$1 + 3\lambda = 5 + 2\alpha \quad (3)$$

$$\text{From (1) } \lambda = 2\alpha$$

$$\text{Sub in (2) } 1 + 4\mu = 4 + \mu$$

$$\therefore 3\alpha = 3 \therefore \alpha = 1, \lambda = 2$$

$$\text{Check in (3) } 1 + 3(2) = 5 + 2(1)$$

$$7 = 7$$

$$\therefore L_1 \text{ and } L_2 \text{ are concurrent}$$

$$\text{point of intersection is } (3, 5, 7)$$

$$\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \beta(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$\text{7 } x + y + z = 3 \quad (1) \quad (2) - (1) \quad x - 3z = -3 \quad (4)$$

$$2x + y - 2z = 0 \quad (2) \quad (3) + 2(1) \quad 5x + 7z = 29 \quad (5)$$

$$3x - 2y + 5z = 23 \quad (3)$$

$$(5) - 5(4) \quad 22z = 44$$

$$\therefore z = 2, x = 3$$

$$y = 2z - 2x = -2$$

$$\therefore x = 3, y = -2, z = 2$$

$(3, -2, 2)$ is the point of intersection of the 3 planes represented by the equations.

$$\text{8 a } 3x + y + z = 1 \quad (1) \quad (1) - (2) \quad 2x + 2z = -3 \quad (4)$$

$$x + y - z = 4 \quad (2) \quad (1) - (3)$$

$$x + (1 - b)z = 1 - a \quad (5)$$

$$2x + y + bz = a \quad (3)$$

$$(4) - 2(5) \quad 2z - 2(1 - b)z = -3 - 2(1 - a)$$

$$2bz = -5 + 2a$$

$$z = \frac{2a - 5}{2b}$$

$$2x = -3 - 2z = -3 - \frac{(2a - 5)}{b} = \frac{-3b - 2a + 5}{b}$$

$$x = \frac{5 - 2a - 3b}{2b}$$

$$y = 4 - x + z = \frac{8b - (5 - 2a - 3b) + 2a - 5}{2b}$$

$$y = \frac{-10 + 4a + 11b}{2b}$$

$$x = \frac{5 - 2a - 3b}{2b}, y = \frac{4a + 11b - 10}{2b}, z = \frac{2a - 5}{2b}$$

$$\text{b } b = 0 \text{ equations (4) and (5) become}$$

$$2x + 2z = -3$$

$$x + z = 1 - a$$

$$\text{For a non-unique solution, } 1 - a = -\frac{3}{2}$$

$$\therefore a = \frac{5}{2}$$

$a = \frac{5}{2}, b = 0$ the planes represented by the 3 equations intersect in a line.

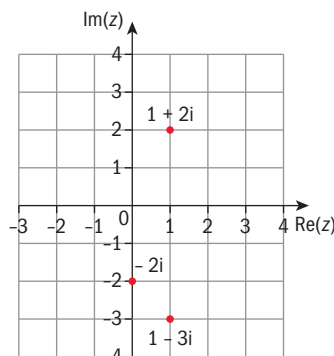
12

Multiple perspectives in mathematics

Answers

Skills check

1



2 $z = 5 - 4i$

a $z^* = 5 + 4i$ $-z = -5 + 4i$

b $\frac{1}{z} = \frac{1}{5-4i} \times \frac{5+4i}{5+4i}$
 $= \frac{5}{41} + \frac{4}{41}i$

3 $z = -3 + 4i$

a $\operatorname{Re}(z) = -3$

b $\operatorname{Im}(z) = 4$

c $|z| = \sqrt{(-3)^2 + 4^2} = 5$

4 $z_1 = 1 - 2i$ $z_2 = 3 + i$

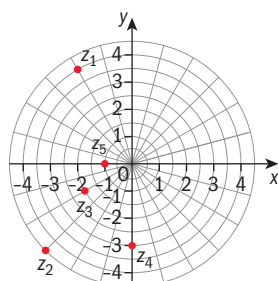
a $z_1 + 2z_2 = 1 - 2i + 6 + 2i = 7$

b $3z_1 z_2 + z_1^2 = 3(1 - 2i)(3 + i) + (1 - 2i)^2$
 $= 3(3 + i - 6i + 2) + (1 - 4i - 4)$
 $= 12 - 19i$

c $\frac{z_1}{z_2} = \frac{1-2i}{3+i} \times \frac{3-i}{3-i} = \frac{3-i-6i-2}{10} = \frac{1}{10} - \frac{7}{10}i$

Exercise 12A

1



$$z_1 = z_2 \Rightarrow |z_1| = |z_2|$$

$$r^3 = r^2 + 2r$$

$$r(r^2 - r - 2) = 0$$

$$r(r-2)(r+1) = 0$$

$$r = 0 \text{ or } 2 \ (r \geq 0)$$

$$\arg(z_1) - \arg(z_2) = 2k\pi \quad k \in \mathbb{Z}$$

$$4\theta - (\theta + \frac{\pi}{2}) = 2k\pi$$

$$3\theta = \frac{\pi}{2} + 2k\pi$$

$$\theta = \frac{\pi}{6} + \frac{2k\pi}{3} \quad k \in \mathbb{Z}$$

Complex numbers are given by $z_1 = z_2 = 0$

or $z_1 = z_2 = 8 \operatorname{cis} \frac{2\pi}{8}$

$$z_1 = z_2 = 8 \operatorname{cis} \frac{4\pi}{3}$$

$$z_1 = z_2 = 8 \operatorname{cis} 2\pi$$

i.e. $0, -4 + 4i\sqrt{3}, -4 - 4i\sqrt{3}, 8$

3 $|a + ai| = 2 \Rightarrow \sqrt{a^2 + a^2} = 2$

$$2a^2 = 4$$

$$a^2 = 2$$

$$a = \pm\sqrt{2}$$

$$\arg(a + ai) = \theta \quad \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

2 distinct complex numbers, $\sqrt{2} + \sqrt{2}i$

and $-\sqrt{2} - \sqrt{2}i$

4 a eg. $z_1 = 3 + 3i$ $z_2 = 4 + 4i$

$$z_1 + z_2 = 7 + 7i \quad z_1 - z_2 = -1 - i$$

$$\arg(z_1) = \frac{\pi}{4} \quad \arg(z_2) = \frac{\pi}{4}$$

$$\arg(z_1 + z_2) = \frac{\pi}{4} \neq \arg(z_1) + \arg(z_2)$$

$$\arg(z_1 - z_2) = \frac{5\pi}{4} \neq \arg(z_1) - \arg(z_2)$$

$$\therefore \arg(z_1 \pm z_2) \neq \arg(z_1) \pm \arg(z_2) \quad (\text{QED})$$

b eg. $z_1 = 3 + 2i$ $z_2 = -1 + 4i$

$$z_1 + z_2 = 2 + 6i \quad z_1 - z_2 = 4 - 2i$$

$$|z_1| = \sqrt{13} \quad |z_2| = \sqrt{17}$$

$$|z_1 + z_2| = \sqrt{40} \neq |z_1| + |z_2|$$

$$|z_1 - z_2| = \sqrt{20} \neq ||z_1| - |z_2||$$

$$\therefore |z_1 \pm z_2| \neq |z_1| \pm |z_2|$$

Exercise 12B

1 a $x = 6\cos 45^\circ = 3\sqrt{2}$ $y = 6\sin 45^\circ = 3\sqrt{2}$

$$z_1 = 3\sqrt{2} + 3\sqrt{2}i$$

b $x = 10\cos 135^\circ = -5\sqrt{2}$ $y = 10\sin 135^\circ = 5\sqrt{2}$

$$z_2 = -5\sqrt{2} + 5\sqrt{2}i$$

c $z_3 = 4 \operatorname{cis} \frac{5\pi}{3} = 4(\cos \frac{5\pi}{3} + i\sin \frac{5\pi}{3})$

$$= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2 - 2\sqrt{3}i$$

$$\begin{aligned} \text{d } z_4 &= 5 \operatorname{cis} \frac{7\pi}{6} = 5\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right) \\ &= 5\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{5\sqrt{3}}{2} - \frac{5}{2}i \end{aligned}$$

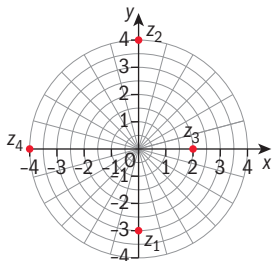
$$2 \quad \text{a } z_1 = -1 - i \quad r = \sqrt{2} \quad \theta = \frac{5\pi}{4} \quad z_1 = \sqrt{2} \operatorname{cis} \frac{5\pi}{4}$$

$$\text{b } z_2 = 2\sqrt{3} + 2i \quad r = 4 \quad \theta = \frac{\pi}{6} \quad z_2 = 4 \operatorname{cis} \frac{\pi}{6}$$

$$\text{c } z_3 = 4 - 4i \quad r = 4\sqrt{2} \quad \theta = \frac{7\pi}{4} \quad z_3 = 4\sqrt{2} \operatorname{cis} \frac{7\pi}{4}$$

$$\text{d } z_4 = -5 + 5i \quad r = 5\sqrt{2} \quad \theta = \frac{3\pi}{4} \quad z_4 = 5\sqrt{2} \operatorname{cis} \frac{3\pi}{4}$$

3



$$\text{a } z_1 = -3i = 3 \operatorname{cis} \frac{3\pi}{2}$$

$$\text{b } z_2 = 4i = 4 \operatorname{cis} \frac{\pi}{2}$$

$$\text{c } z_3 = 2 = 2 \operatorname{cis} 0$$

$$\text{d } z_4 = -4 = 4 \operatorname{cis} \pi$$

$$4 \quad z = 4 \operatorname{cis} 40^\circ$$

$$\text{a } z^* = 4 \operatorname{cis} (-40^\circ) \text{ or } 4 \operatorname{cis} 320^\circ$$

$$\text{b } -z = 4 \operatorname{cis} 220^\circ$$

$$\text{c } -z^* = 4 \operatorname{cis} 140^\circ$$

$$\text{d } 3z^* = 12 \operatorname{cis} (-40^\circ) \text{ or } 12 \operatorname{cis} 320^\circ$$

$$\text{e } -4z^* = 16 \operatorname{cis} 140^\circ$$

$$5 \quad z_1 = -2 - 2\sqrt{3}i \quad z_2 = 3\sqrt{3} + 3i$$

$$\begin{aligned} \text{a } z_3 &= (-2 - 2\sqrt{3}i)(3\sqrt{3} + 3i) \\ &= -6\sqrt{3} - 6i - 18i + 6\sqrt{3} \\ &= -24i \end{aligned}$$

$$\text{b } |z_1| = \sqrt{4 + 12} = 4 \quad \arg(z_1) = \frac{4\pi}{3} \quad z_1 = 4 \operatorname{cis} \frac{4\pi}{3}$$

$$|z_2| = \sqrt{27 + 9} = 6 \quad \arg(z_2) = \frac{\pi}{6} \quad z_2 = 6 \operatorname{cis} \frac{\pi}{6}$$

$$|z_3| = 24 \quad \arg(z_3) = \frac{3\pi}{2} \quad z_3 = 24 \operatorname{cis} \left(\frac{3\pi}{2}\right)$$

$$\text{c } |z_3| = |z_1| |z_2|, \quad \arg(z_3) = \arg(z_1) + \arg(z_2)$$

Exercise 12C

$$1 \quad \text{a } z_1 z_2 = 5 \operatorname{cis} 135^\circ$$

$$\text{b } z_1 z_2 = \frac{2}{7} \operatorname{cis} \frac{31\pi}{24}$$

$$2 \quad z_1 = \operatorname{cis} \frac{5\pi}{6} \quad z_2 = 1 - i$$

$$\text{a } z_1 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\text{b } z_2 = \sqrt{2} \operatorname{cis} \frac{7\pi}{4}$$

$$\begin{aligned} \text{c } z_1 z_2 &= \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)(1 - i) = -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2}i + \frac{1}{2} \\ &= -\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i \end{aligned}$$

$$z_1 z_2 = \sqrt{2} \operatorname{cis} \left(\frac{5\pi}{6} + \frac{7\pi}{4}\right) = \sqrt{2} \operatorname{cis} \frac{31\pi}{12} = \sqrt{2} \operatorname{cis} \frac{7\pi}{12}$$

$$\sqrt{2} \sin \frac{7\pi}{12} = \frac{\sqrt{3}+1}{2} \therefore \sin \frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\sqrt{2} \cos \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{2} \therefore \cos \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{2\sqrt{2}} = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\tan \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

$$3 \quad z_1 = 2 \operatorname{cis} \frac{\pi}{6} \quad z_2 = r \operatorname{cis} \theta \quad r > 0, 0 < \theta < 2\pi$$

$$\text{a } z_1 z_2 = 2r \operatorname{cis} \left(\frac{\pi}{6} + \theta\right)$$

$$z_1 z_2 \text{ real} \Rightarrow \sin \left(\frac{\pi}{6} + \theta\right) = 0$$

$$\frac{\pi}{6} + \theta = \pi \text{ or } 2\pi \quad \theta = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

$$z_1 z_2 > 5 \Rightarrow \theta = \frac{11\pi}{6} \text{ and } 2r > 5 \Rightarrow r > \frac{5}{2}$$

$$r > \frac{5}{2} \text{ and } \theta = \frac{11\pi}{6}$$

$$\text{b } z_1 z_2 \text{ imaginary} \Rightarrow \cos \left(\frac{\pi}{6} + \theta\right) = 0$$

$$\frac{\pi}{6} + \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\theta = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$$

$$|z_1 z_2| < 1 \Rightarrow 2r < 1 \Rightarrow r < \frac{1}{2}$$

$$0 < r < \frac{1}{2} \text{ and } \theta = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$$

$$(r \neq 0 \text{ otherwise } z_1 z_2 \text{ is real})$$

$$4 \quad z = -3 + \sqrt{3}i = 2\sqrt{3} \operatorname{cis} \frac{5\pi}{6}$$

$$-z = z\sqrt{3} \operatorname{cis} \frac{11\pi}{6}, z^* = 2\sqrt{3} \operatorname{cis} \left(-\frac{5\pi}{6}\right) \text{ or } 2\sqrt{3} \operatorname{cis} \frac{7\pi}{6}$$

$$-z^* = 2\sqrt{3} \operatorname{cis} \frac{\pi}{6}$$

$$5 \quad z = \sin \alpha + \cos \alpha i \quad w = \sin 2\alpha - \cos 2\alpha i$$

$$\text{a } |z| = 1 \quad \tan \theta = \cot \alpha \therefore \theta = \frac{\pi}{2} - \alpha,$$

$$z = \operatorname{cis} \left(\frac{\pi}{2} - \alpha\right)$$

$$|w| = 1 \quad \tan \theta = -\cot 2\alpha$$

$$\theta = -\left(\frac{\pi}{2} - 2\alpha\right) \text{ or } 2\pi - \left(\frac{\pi}{2} - 2\alpha\right)$$

$$\theta = \frac{3\pi}{2} + 2\alpha \therefore w = \operatorname{cis} \left(\frac{3\pi}{2} + 2\alpha\right)$$

$$\text{b } |zw| = 1 \quad \arg(zw) = \frac{\pi}{2} - \alpha + \frac{3\pi}{2} + 2\alpha$$

$$= 2\pi + \alpha \text{ or } \alpha$$

$$zw = \operatorname{cis} \alpha$$

Exercise 12D

$$1 \quad \text{a } \frac{z_1}{z_2} = 2 \operatorname{cis} 45^\circ$$

$$\text{b } \frac{z_1}{z_2} = \frac{7}{8} \operatorname{cis} \frac{25\pi}{24}$$

$$2 \quad z_1 = 2 \operatorname{cis} \frac{11\pi}{6} \quad z_2 = 2 - 2i$$

$$\text{a } |z_2| = 2\sqrt{2} \quad \arg(2 - 2i) = \frac{7\pi}{4} \quad z_2 = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{4}$$

$$\text{b } z_2^* = 2\sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

c $z_1 z_2 = 4\sqrt{2} \operatorname{cis} \frac{43\pi}{12} = 4\sqrt{2} \operatorname{cis} \frac{19\pi}{12}$

d $\frac{z_1}{z_2} = \frac{1}{\sqrt{2}} \operatorname{cis} \frac{\pi}{12} = \frac{\sqrt{2}}{2} \operatorname{cis} \frac{\pi}{12}$

e $\frac{1}{z_1 z_2} = \frac{1}{4\sqrt{2}} \operatorname{cis} \left(-\frac{19\pi}{12}\right) = \frac{\sqrt{2}}{8} \operatorname{cis} \frac{5\pi}{12}$
 $\therefore -\frac{1}{z_1 z_2} = \frac{\sqrt{2}}{8} \operatorname{cis} \frac{17\pi}{12}$

3 a $4 = 4 \operatorname{cis} 0 \quad \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$
 $\frac{4}{\sqrt{3} + i} = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right) = 2 \operatorname{cis} \frac{11\pi}{6}$

b $2 - 2i = 2\sqrt{2} \operatorname{cis} \frac{7\pi}{4} \quad \sqrt{6} + \sqrt{2}i = 2\sqrt{2} \operatorname{cis} \frac{\pi}{6}$
 $\frac{2 - 2i}{\sqrt{6} + \sqrt{2}i} = \operatorname{cis} \frac{19\pi}{12}$

c $1 = \operatorname{cis} 0 \quad (\sqrt{21} - \sqrt{7}i)^* = \sqrt{21} + \sqrt{7}i = 2\sqrt{7} \operatorname{cis} \frac{\pi}{6}$
 $\frac{1}{(\sqrt{21} - \sqrt{7}i)^*} = \frac{1}{2\sqrt{7}} \operatorname{cis} \left(-\frac{\pi}{6}\right) = \frac{\sqrt{7}}{14} \operatorname{cis} \frac{11\pi}{6}$

4 $z = 2\sqrt{3} - 2i \quad w = \frac{1-i}{2}$

a $|z| = 4 \arg(z) = -\frac{\pi}{6} \quad z = 4 \left(\cos \left(-\frac{\pi}{6}\right) + i \sin \left(-\frac{\pi}{6}\right) \right)$
 $|w| = \frac{\sqrt{2}}{2} \arg(w) = -\frac{\pi}{4}$
 $w = \frac{\sqrt{2}}{2} \left(\cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right)$

b $\frac{z}{w} = \frac{8}{\sqrt{2}} \operatorname{cis} \frac{\pi}{12} = 4\sqrt{2} \operatorname{cis} \frac{\pi}{12}$

c $\frac{z}{w} = \frac{2\sqrt{3} - 2i}{\frac{1-i}{2}} = \frac{4\sqrt{3} - 4i}{1-i} \times \frac{1+i}{1+i}$
 $= \frac{4\sqrt{3} + 4\sqrt{3}i - 4i + 4}{2}$
 $= (2\sqrt{3} + 2) + (2\sqrt{3} - 2)i$

d $4\sqrt{2} \cos \frac{\pi}{12} = 2\sqrt{3} + 2$
 $\therefore \cos \frac{\pi}{12} = \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4}$
 $4\sqrt{2} \sin \frac{\pi}{12} = 2\sqrt{3} - 2$
 $\therefore \sin \frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$
 $\tan \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} \times \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{6 - 2\sqrt{12} + 2}{4}$
 $\tan \frac{\pi}{12} = 2 - \sqrt{3}$

Exercise 12E

1 $z_1 = z \operatorname{cis} \frac{3\pi}{4} \quad z_2 = \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$

a $z_1^* = 2 \operatorname{cis} \frac{5\pi}{4} \quad (z_1^*)^2 = 4 \operatorname{cis} \frac{5\pi}{2} = 4 \operatorname{cis} \frac{\pi}{2}$
 $z_2^3 = 8 \operatorname{cis} \frac{\pi}{2} \therefore (z_1^*)^2 (z_2)^3 = 32 \operatorname{cis} \pi \text{ or } -32$

b $\frac{z_1}{z_2} = \operatorname{cis} \frac{7\pi}{12} \therefore \left(\frac{z_1}{z_2}\right)^4 = \operatorname{cis} \frac{\pi}{3} = \operatorname{cis} \frac{\pi}{3}$

c $\frac{z_1^*}{z_2} = \operatorname{cis} \frac{\pi}{12} \therefore \left(\frac{z_1^*}{z_2}\right)^3 = \operatorname{cis} \frac{3\pi}{12} = \operatorname{cis} \frac{3\pi}{4}$

2 $z_1 = 4e^{\frac{\pi i}{4}} \quad z_2 = \frac{1}{2}e^{\frac{\pi i}{3}}$

a $z_1 z_2 = 2e^{\frac{\pi i}{12}}$

b $(z_1)^3 = 64e^{\frac{3\pi i}{4}} \quad (z_2)^{-2} = 4e^{\frac{2\pi i}{3}}$
 $(z_1)^3 (z_2)^{-2} = 256e^{\frac{17\pi i}{12}} = 256e^{\frac{7\pi i}{12}}$

c $\frac{z_1}{z_2} = 8e^{\frac{7\pi i}{12}}$

d $z_1^* = 4e^{\frac{\pi i}{4}} \therefore \frac{z_1^*}{z_2} = 8e^{\frac{\pi i}{2}} \therefore \left(\frac{z_1^*}{z_2}\right)^{-3} = \frac{1}{512}e^{\frac{\pi i}{4}}$

3 $\left(\frac{\cos \theta - i \sin \theta}{\sin \theta + i \cos \theta}\right)^5 = \left(\frac{\cos(-\theta) + i \sin(-\theta)}{\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right)}\right)^5$
 $= [\operatorname{cis}(-\theta - (\frac{\pi}{2} - \theta))]^5$
 $= \left(\operatorname{cis}\left(-\frac{\pi}{2}\right)\right)^5 = (-i)^5 = -i$

4 $1 + i\sqrt{3} = 2 \operatorname{cis} \frac{\pi}{3} \quad 1 - i\sqrt{3} = 2 \operatorname{cis} \left(-\frac{\pi}{3}\right)$
 $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n = 2^n \operatorname{cis} \frac{n\pi}{3} + 2^n \operatorname{cis} \left(-\frac{n\pi}{3}\right)$
 $= 2^n [\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} + \cos \left(-\frac{n\pi}{3}\right) + i \sin \left(-\frac{n\pi}{3}\right)]$
 $= 2^n (\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} + \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3})$
 $= 2^n (2 \cos \frac{n\pi}{3}) = 2^{n+1} \cos \frac{n\pi}{3}, n \in \mathbb{N} \text{ (QED)}$

5 a $(1 - i)^n = [\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right)]^n = 2^{\frac{n}{2}} \operatorname{cis} \left(-\frac{n\pi}{4}\right)$
 $= 2^{\frac{n}{2}} (\cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4})$
 $\sin \frac{n\pi}{4} = 0 \Rightarrow \frac{n\pi}{4} = \pi, 2\pi, 3\pi, \dots$
 $\Rightarrow n = 4, 8, 12, \dots$
 $\cos \frac{n\pi}{4} < 0 \therefore n = 4$

b $\cos \frac{n\pi}{4} = 0 \quad \frac{n\pi}{4} = \frac{\pi}{2} \therefore n = 2$

6 Let $z = r \operatorname{cis} \theta \quad r(r \operatorname{cis} \theta)^3 = 16$
 $r^4 (\cos 3\theta + i \sin 3\theta) = 16$
 $\left. \begin{aligned} \sin 3\theta = 0 &\Rightarrow 3\theta = k\pi \Rightarrow \theta = \frac{k\pi}{3} \quad k \in \mathbb{Z} \\ \cos 3\theta = 1 &\Rightarrow 3\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{3} \quad k \in \mathbb{Z} \end{aligned} \right\}$
 $\therefore \theta = \frac{2k\pi}{3}, r^4 = 16 \Rightarrow r = 2$
 $z = 2 \operatorname{cis} \left(\frac{2k\pi}{3}\right) \text{ (i.e. } z = 2 \operatorname{cis} 0, 2 \operatorname{cis} \frac{2\pi}{3}, 2 \operatorname{cis} \frac{4\pi}{3})$
 $z = 2(1 + 0i), 2(-\frac{1}{2} + \frac{\sqrt{3}}{2}i), 2(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$
 $z = 2, -1 + \sqrt{3}i, -1 - \sqrt{3}i$

Exercise 12F

1 a $z_3 = 8 \operatorname{cis} \frac{\pi}{3} \Rightarrow z = 2 \operatorname{cis} \left(\frac{\pi}{9} + \frac{2k\pi}{3}\right) k = 0, 1, 2$
 $z = 2 \operatorname{cis} \frac{\pi}{9}, 2 \operatorname{cis} \frac{7\pi}{9}, 2 \operatorname{cis} \frac{13\pi}{9}$

b $z^4 = (-4i)^2 = -16 = 16 \operatorname{cis} \pi$
 $z = 2 \operatorname{cis} \left(\frac{\pi + 2k\pi}{4}\right) \quad k = 0, 1, 2, 3$

$$z = 2 \operatorname{cis} \frac{\pi}{4}, 2 \operatorname{cis} \frac{3\pi}{4}, 2 \operatorname{cis} \frac{5\pi}{4}, 2 \operatorname{cis} \frac{7\pi}{4}$$

$$z = 2 \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right), 2 \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right), 2 \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right), 2 \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)$$

$$z = \sqrt{2} + i\sqrt{2}, -\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}, \sqrt{2} - i\sqrt{2}$$

$$\mathbf{c} \quad z^5 = 32e^{-\pi i} \Rightarrow z = 2e^{\frac{-\pi + 2k\pi}{5}i} \quad k = 0, 1, 2, 3, 4$$

$$z = 2e^{\frac{\pi i}{5}}, 2e^{\frac{3\pi i}{5}}, 2e^{\frac{5\pi i}{5}}, 2e^{\frac{7\pi i}{5}}, 2e^{\frac{9\pi i}{5}}$$

$$\mathbf{2} \quad \mathbf{a} \quad z^2 = 1 - i = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

$$z = 2^{\frac{1}{4}} \operatorname{cis} \left(-\frac{\pi}{8} + \frac{2k\pi}{2} \right) \quad k = 0, 1$$

$$z = 2^{\frac{1}{4}} \operatorname{cis} \left(-\frac{\pi}{8} \right), 2^{\frac{1}{4}} \operatorname{cis} \frac{7\pi}{8}$$

$$\mathbf{b} \quad z^4 = -\sqrt{3} + i = 2 \operatorname{cis} \frac{5\pi}{6}$$

$$z = 2^{\frac{1}{4}} \operatorname{cis} \left(\frac{5\pi}{24} + \frac{2k\pi}{4} \right) \quad k = 0, 1, 2, 3$$

$$z = 2^{\frac{1}{4}} \operatorname{cis} \frac{5\pi}{24}, 2^{\frac{1}{4}} \operatorname{cis} \frac{17\pi}{24}, 2^{\frac{1}{4}} \operatorname{cis} \frac{29\pi}{24}, 2^{\frac{1}{4}} \operatorname{cis} \frac{41\pi}{24}$$

$$\mathbf{c} \quad z^3 = 27e^{\frac{\pi i}{4}} \Rightarrow z = 3e^{\left(\frac{\pi + 2k\pi}{12} \right)i} \quad k = 0, 1, 2$$

$$z = 3e^{\frac{\pi i}{12}}, 3e^{\frac{3\pi i}{4}}, 3e^{\frac{17\pi i}{12}}$$

$$\mathbf{d} \quad z^3 - (\sqrt{2} - \sqrt{2}i)z = 0$$

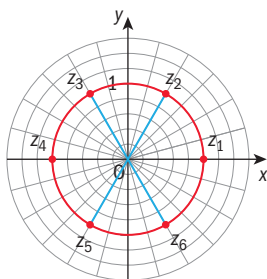
$$z = 0 \text{ or } z^2 = \sqrt{2} - \sqrt{2}i = 2 \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

$$z = 0 \text{ or } z = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{8} + \frac{2k\pi}{2} \right) \quad k = 0, 1$$

$$z = 0, \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{8} \right), \sqrt{2} \operatorname{cis} \frac{7\pi}{8}$$

$$\mathbf{3} \quad \mathbf{a} \quad z^6 = 1 = \operatorname{cis} 0 \Rightarrow z = \operatorname{cis} \frac{2k\pi}{6}, k = 0, 1, 2, 3, 4, 5$$

$$z = 1, \operatorname{cis} \frac{\pi}{3}, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \pi, \operatorname{cis} \frac{4\pi}{3}, \operatorname{cis} \frac{5\pi}{3}$$



$$\mathbf{b} \quad z_1^3 = 1^3 = 1, z_3^3 = \left(\operatorname{cis} \frac{2\pi}{3} \right)^3 = \operatorname{cis} 2\pi = 1$$

$$z_5^3 = \left(\operatorname{cis} \frac{4\pi}{3} \right)^3 = \operatorname{cis} 4\pi = 1$$

$$\therefore z_1^3 = z_3^3 = z_5^3 \quad (\text{QED})$$

$$z_2^3 = \left(\operatorname{cis} \frac{\pi}{3} \right)^3 = \operatorname{cis} \pi = -1, z_4^3 = \left(\operatorname{cis} \pi \right)^3 = \operatorname{cis} 3\pi = -1$$

$$z_6^3 = \left(\operatorname{cis} \frac{5\pi}{3} \right)^3 = \operatorname{cis} 5\pi = -1$$

$$\therefore z_2^3 = z_4^3 = z_6^3 \quad (\text{QED})$$

z_1, z_3, z_5 are the cube roots of 1 while z_2, z_4, z_6 are the cube roots of -1.

$$\mathbf{4} \quad \frac{-1 + i\sqrt{3}}{4} = \frac{1}{2} \operatorname{cis} \frac{2\pi}{3}$$

$$\text{the other roots are } \frac{1}{2} \operatorname{cis} \left(\frac{2\pi}{3} + \frac{2k\pi}{3} \right) \quad k = 1, 2, 3, 4$$

$$\text{ie. } \frac{1}{2} \operatorname{cis} \frac{16\pi}{15}, \frac{1}{2} \operatorname{cis} \frac{22\pi}{15}, \frac{1}{2} \operatorname{cis} \frac{28\pi}{15}, \frac{1}{2} \operatorname{cis} \frac{34\pi}{15}$$

$$\mathbf{5} \quad z^4 = -81 = 81 \operatorname{cis} \pi \Rightarrow z = 3 \operatorname{cis} \left(\frac{\pi}{4} + \frac{2k\pi}{4} \right) \quad k = 0, 1, 2, 3$$

$$z = 3 \operatorname{cis} \frac{\pi}{4}, 3 \operatorname{cis} \frac{3\pi}{4}, 3 \operatorname{cis} \frac{5\pi}{4}, 3 \operatorname{cis} \frac{7\pi}{4}$$

$$z = \frac{3}{2} \sqrt{2} + \frac{3}{2} \sqrt{2}i, -\frac{3}{2} \sqrt{2} + \frac{3}{2} \sqrt{2}i, -\frac{3}{2} \sqrt{2} - \frac{3}{2} \sqrt{2}i, \frac{3}{2} \sqrt{2} - \frac{3}{2} \sqrt{2}i$$

$$(z - 3)^4 + 81 = 0 \Rightarrow (z - 3)^4 = -81$$

$$\therefore z = \left(\frac{3}{2} \sqrt{2} + 3 \right) + \frac{3}{2} \sqrt{2}i, \left(-\frac{3}{2} \sqrt{2} + 3 \right) + \frac{3}{2} \sqrt{2}i,$$

$$\left(-\frac{3}{2} \sqrt{2} + 3 \right) - \frac{3}{2} \sqrt{2}i, \left(\frac{3}{2} \sqrt{2} + 3 \right) - \frac{3}{2} \sqrt{2}i$$

$$\mathbf{6} \quad e^{0i\theta} + \frac{1}{3} e^{2i\theta} + \frac{1}{9} e^{4i\theta} + \frac{1}{27} e^{6i\theta} + \dots$$

$$= \frac{1}{1 - \frac{1}{3} e^{2i\theta}} = \frac{3}{3 - e^{2i\theta}}$$

Exercise 12G

$$\mathbf{1} \quad z = \operatorname{cis} \alpha$$

$$\mathbf{a} \quad z^n = \cos n\alpha + i \sin n\alpha \left(\frac{1}{z} \right)^n = (\operatorname{cis} (-\alpha))^n$$

$$= \operatorname{cis} (-n\alpha)$$

$$= \cos(-n\alpha) + i \sin(-n\alpha)$$

$$= \cos n\alpha - i \sin n\alpha$$

$$z^n - \left(\frac{1}{z} \right)^n = \cos n\alpha + i \sin n\alpha - (\cos n\alpha - i \sin n\alpha)$$

$$= 2i \sin(n\alpha) \quad (\text{QED})$$

$$\mathbf{b} \quad \left(z - \frac{1}{z} \right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$(2i \sin \alpha)^5 = \left(z^5 - \frac{1}{z^5} \right) - 5 \left(z^3 - \frac{1}{z^3} \right) + 10 \left(z - \frac{1}{z} \right)$$

$$32i \sin^5 \alpha = 2i \sin 5\alpha - 10i \sin 3\alpha +$$

$$20i \sin \alpha \quad (\div 2i)$$

$$16 \sin^5 \alpha = \sin 5\alpha - 5 \sin 3\alpha + 10 \sin \alpha$$

$$\int \sin^5 \alpha \, d\alpha = \frac{1}{16} \int (\sin 5\alpha - 5 \sin 3\alpha + 10 \sin \alpha) \, d\alpha$$

$$= \frac{1}{16} \left(-\frac{1}{5} \cos 5\alpha + \frac{5}{3} \sin 3\alpha - 10 \cos \alpha \right) + c$$

$$= -\frac{1}{80} \cos 5\alpha + \frac{5}{48} \sin 3\alpha - \frac{5}{8} \cos \alpha + c$$

$$\mathbf{2} \quad (\cos \alpha + i \sin \alpha)^5 = \cos^5 \alpha + 5i \cos^4 \alpha \sin \alpha$$

$$+ 10i^2 \cos^3 \alpha \sin^2 \alpha$$

$$+ 10i^3 \cos^2 \alpha \sin^3 \alpha$$

$$+ 5i^4 \cos \alpha \sin^4 \alpha + i^5 \sin^5 \alpha$$

$$\therefore \cos^5 \alpha + i \sin^5 \alpha = \cos^5 \alpha + 5i \cos^4 \alpha \sin \alpha$$

$$- 10 \cos^3 \alpha \sin^2 \alpha$$

$$- 10i \cos^2 \alpha \sin^3 \alpha$$

$$+ 5 \cos \alpha \sin^4 \alpha + i \sin^5 \alpha$$

Equating real and imaginary parts:

$$\begin{aligned}\cos 5\alpha &= \cos^5 \alpha - 10\cos^3 \alpha \sin^2 \alpha + 5\cos \alpha \sin^4 \alpha \\ \sin 5\alpha &= 5\cos^4 \alpha \sin \alpha - 10\cos^2 \alpha \sin^3 \alpha + \sin^5 \alpha \\ \tan 5\alpha &= \frac{\sin 5\alpha}{\cos 5\alpha} \\ &= \frac{5\cos^4 \alpha \sin \alpha - 10\cos^2 \alpha \sin^3 \alpha + \sin^5 \alpha}{\cos^5 \alpha - 10\cos^3 \alpha \sin^2 \alpha + 5\cos \alpha \sin^4 \alpha}\end{aligned}$$

Dividing top and bottom by $\cos^5 \alpha$:

$$\tan 5\alpha = \frac{5\tan \alpha - 10\tan^3 \alpha + \tan^5 \alpha}{1 - 10\tan^2 \alpha + 5\tan^4 \alpha} \quad (\text{QED})$$

$$\text{Let } 5\alpha = \pi, \alpha = \frac{\pi}{5}$$

$$\tan \pi = 0 \therefore 5 \tan \frac{\pi}{5} - 10 \tan^3 \frac{\pi}{5} + \tan^5 \frac{\pi}{5} = 0$$

$$\tan \frac{\pi}{5} \neq 0 \therefore 5 - 10 \tan^2 \frac{\pi}{5} + \tan^4 \frac{\pi}{5} = 0$$

$$\tan^2 \frac{\pi}{5} = \frac{10 \pm \sqrt{100 - 20}}{2}$$

$$\tan^2 \frac{\pi}{5} = 5 \pm 2\sqrt{5}$$

$$\tan \frac{\pi}{5} < 1 \therefore \tan^2 \frac{\pi}{5} = 5 - 2\sqrt{5}$$

$$\therefore \tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}} \quad (\text{QED})$$

$$\begin{aligned}3 \quad (\cos \theta + i \sin \theta)^7 &= \cos^7 \theta + 7\cos^6 \theta i \sin \theta \\ &+ 21 \cos^5 \theta i^2 \sin^2 \theta + 35 \cos^4 \theta i^3 \sin^3 \theta \\ &+ 35 \cos^3 \theta i^4 \sin^4 \theta + 21 \cos^2 \theta i^5 \sin^5 \theta \\ &+ 7 \cos \theta i^6 \sin^6 \theta + i^7 \sin^7 \theta \\ &= \cos^7 \theta + 7i \cos^6 \theta \sin \theta - 21 \cos^5 \theta \sin^2 \theta \\ &- 35i \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta + 21i \cos^2 \theta \sin^5 \theta \\ &- 7 \cos \theta \sin^6 \theta - i \sin^7 \theta\end{aligned}$$

Equating real parts,

$$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$$

$$\begin{aligned}\cos 7\theta &= \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta) \\ &+ 35 \cos^3 \theta (1 - \cos^2 \theta)^2 - 7 \cos \theta (1 - \cos^2 \theta)^3 \\ &= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta \\ &\quad (1 - 2\cos^2 \theta + \cos^4 \theta) - 7 \cos \theta \\ &\quad (1 - 3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta) \\ &= 22 \cos^7 \theta - 21 \cos^5 \theta + 35 \cos^3 \theta - 7 \cos \theta \\ &\quad \cos^5 \theta + 35 \cos^7 \theta - 7 \cos \theta + 21 \cos^3 \theta \\ &\quad - 21 \cos^5 \theta + 7 \cos^7 \theta\end{aligned}$$

$$\therefore \cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta \quad (\text{QED})$$

$$64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta = 1$$

$$\therefore \cos 7\theta = 1$$

$$7\theta = 2k\pi \quad k \in \mathbb{Z}$$

$$\therefore \theta = \frac{2k\pi}{7} \quad k \in \mathbb{Z}$$

$$4 \quad z = \cos^2 \theta + \frac{\sin 2\theta}{2}i - \frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\begin{aligned}a \quad |z|^2 &= \cos^4 \theta + \frac{\sin^2 2\theta}{2} = \cos^4 \theta + \frac{4\sin^2 \theta \cos^2 \theta}{4} \\ &= \cos^4 \theta + \sin^2 \theta \cos^2 \theta = \cos^2 \theta (\cos^2 \theta + \sin^2 \theta) \\ &= \cos^2 \theta\end{aligned}$$

$$\therefore |z| = \cos \theta \quad (\text{QED})$$

$$\begin{aligned}\arg z &= \tan^{-1} \left(\frac{\sin 2\theta}{2\cos^2 \theta} \right) = \tan^{-1} \left(\frac{2\sin \theta \cos \theta}{2\cos^2 \theta} \right) \\ &= \tan^{-1} \left(\frac{\sin \theta}{\cos \theta} \right) = \tan^{-1}(\tan \theta) = \theta \quad (\text{QED})\end{aligned}$$

$$b \quad z = \cos \theta \operatorname{cis} \theta$$

$$z^2 = \cos^2 \theta \operatorname{cis} 2\theta$$

$$c \quad |2z^2| = |z| \Rightarrow 2\cos^2 \theta = \cos \theta$$

$$\cos \theta (2 \cos \theta - 1) = 0$$

$$\cos \theta = 0 \text{ or } \frac{1}{2}$$

$$\theta = -\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$$

$$5 \quad w = \frac{z+i}{z+2} \quad z = x + iy$$

$$\begin{aligned}a \quad w &= \frac{x+i(y+1)}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy} \\ &= \frac{x(x+2) + y(y+1) + i(y+1)(x+2) - ixy}{(x+2)^2 + y^2}\end{aligned}$$

$$\operatorname{Re}(w) = \frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2}, \quad \operatorname{Im}(w) = \frac{x+2y+2}{(x+2)^2 + y^2} \quad (\text{QED})$$

$$b \quad i \quad \operatorname{Re}(w) = 1, x^2 + 2x + y^2 + y = x^2 + 4x + 4 + y^2$$

$$y = 2x + 4 \quad (\ell_1)$$

$\therefore$ the points (x, y) lie on a straight line, gradient = 2

$$ii \quad \operatorname{Im}(w) = 0, x + 2y + 2 = 0$$

$$y = -\frac{1}{2}x - 1 \quad (\ell_2)$$

$\therefore$ the points (x, y) lie on a straight line, gradient = $-\frac{1}{2}$

$$-\frac{1}{2} \times 2 = -1 \therefore \ell_1 \text{ and } \ell_2 \text{ are perpendicular}$$

(QED)

$$c \quad \operatorname{Arg}(z) = \operatorname{Arg}(w) = \frac{\pi}{4}$$

$$\therefore \frac{y}{x} = 1 \text{ and } \frac{x+2y+2}{x^2+2x+y^2+y} = 1$$

$$y = x \quad x + 2y + 2 = x^2 + 2x + y^2 + y$$

$$3x + 2 = 2x^2 + 3x$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

$$x = 1, y = 1 \text{ or } x = -1 \therefore |z| = \sqrt{2}$$

$$6 \quad z = r \operatorname{cis} \theta$$

$$a \quad p(n): (z^n)^* = (z^*)^n$$

$$p(1): (z^1)^* = (z^*)^1$$

$$z^* = z^* \therefore p(1) \text{ is true}$$

$$\text{Assume } p(k): (z^k)^* = (z^*)^k$$

$$\text{prove } p(k+1): (z^{k+1})^* = (z^*)^{k+1} \quad z^* = (z^*)^* z^* = (z^k)^* z^*$$

$$\text{Let } z_1 = r, \operatorname{cis} \theta_1 \quad z_2 = r_2 \operatorname{cis} \theta_2$$

$$z_1^* = r_1 \operatorname{cis} (-\theta_1) \quad z_2^* = r_2 \operatorname{cis} (-\theta_2)$$

$$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$$

$$z_1^* z_2^* = r_1 r_2 \operatorname{cis} (-\theta_1 - \theta_2)$$

$$(z_1 z_2)^* = r_1 r_2 \operatorname{cis} [-(\theta_1 + \theta_2)] = r_1 r_2 \operatorname{cis} (-\theta_1 - \theta_2)$$

$$\therefore z_1^* z_2^* = (z_1 z_2)^*$$

$$\therefore (z^*)^{k+1} = (z^k z)^* = (z^{k+1})^*$$

$$\therefore p(k) \Rightarrow p(k+1) \text{ and } p(1) \text{ is true}$$

$$\therefore \text{by induction } (z^n)^* = (z^*)^n \quad n \in \mathbb{Z}^+ \quad (\text{QED})$$

$$\mathbf{b} \quad z = r \operatorname{cis} \theta, z^n = r^n \operatorname{cis} (n\theta) \quad (z^n)^* = r^n \operatorname{cis} (-n\theta)$$

$$z^* = r \operatorname{cis} (-\theta), (z^*)^n = r^n \operatorname{cis} (-n\theta)$$

$$\therefore (z^n)^* = (z^*)^n \quad (\text{QED})$$

Valid for $n \in \mathbb{Z}$ since de Moivre's theorem is valid for $n \in \mathbb{Z}$.

$$\mathbf{7} \quad \mathbf{a} \quad z^3 = 1 = \operatorname{cis} 0$$

$$z = \operatorname{cis} \left(\frac{2k\pi}{3} \right) \quad k = 0, 1, 2$$

$$z = \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3}$$

$$z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\text{Let } w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } w^* = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\frac{1}{1+w} = \frac{1}{1+\frac{\sqrt{3}}{2}i} \times \frac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{\frac{1}{2}-\frac{\sqrt{3}}{2}i} = \frac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{1} = -w$$

$$\therefore \frac{1}{1+w} = -w \quad (\text{QED})$$

$$\frac{1}{1+w^*} = \frac{1}{1-\frac{\sqrt{3}}{2}i} \times \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{1} = -w^*$$

$$\therefore \frac{1}{1+w^*} = -w^* \quad (\text{QED})$$

$$\mathbf{b} \quad 1, -w, -w^* \text{ are zeros of } p(z)$$

$$p(z) = (z-1)(z+w)(z+w^*)$$

$$= (z-1)(z^2 + z(w+w^*) + ww^*)$$

$$w + w^* = -1 \quad ww^* = 1$$

$$\therefore p(z) = (z-1)(z^2 - z + 1)$$

$$= z^3 - 2z^2 + 2z - 1$$

$$\therefore a = -2, b = 2, c = -1$$

$$\mathbf{c} \quad p(w) = w^3 - 2w^2 + 2w - 1$$

$$= 1 - 2w^2 + 2w - 1$$

$$= -2w^2 + 2w$$

$$= 2w(1-w)$$

$$= (-1 + \sqrt{3}i) \left(\frac{3}{2} - \frac{\sqrt{3}}{2}i \right)$$

$$= \frac{1}{2}(-3 + \sqrt{3}i + 3\sqrt{3}i + 3)$$

$$= \frac{1}{2}(4\sqrt{3}i) \therefore p(w) = 2\sqrt{3}i$$

$$p(w^*) = (w^*)^3 - 2(w^*)^2 + 2w^* - 1$$

$$= 1 - 2(w^*)^2 + 2w^* - 1$$

$$= 2w^*(1-w^*)$$

$$= (-1 - \sqrt{3}i) \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i \right)$$

$$= \frac{1}{2}(-3 - \sqrt{3}i - 3\sqrt{3}i + 3)$$

$$= \frac{1}{2}(-4\sqrt{3}i) \therefore p(w^*)$$

$$= -2\sqrt{3}i$$

$$\mathbf{9} \quad (\sqrt{3} - i)^n = \left(2 \operatorname{cis} \left(-\frac{\pi}{6} \right) \right)^n = 2^n \operatorname{cis} \left(-\frac{n\pi}{6} \right)$$

$$\Rightarrow \cos \left(-\frac{n\pi}{6} \right) > 0 \text{ and } \sin \left(-\frac{n\pi}{6} \right) = 0$$

$$\frac{n\pi}{6} = k\pi \quad k \in \mathbb{Z}$$

$$n = 6k \quad k \in \mathbb{Z}$$

$$\cos \left(\frac{n\pi}{6} \right) > 0 \quad n = 12k \quad k \in \mathbb{Z}$$

$$\mathbf{10} \quad f(z) = \ln(|z|) + i \arg(z)$$

$$\mathbf{a} \quad f(i) = \ln 1 + i \frac{\pi}{2} \therefore f(i) = i \frac{\pi}{2}$$

$$f(-i) = \ln 1 - i \frac{\pi}{2} \therefore f(-i) = -i \frac{\pi}{2}$$

$$f(1+i) = \ln \sqrt{2} + i \frac{\pi}{4} \quad f(1-i) = \ln \sqrt{2} - i \frac{\pi}{4}$$

$$\mathbf{b} \quad (f(z))^* = \ln(|z|) - i \arg(z)$$

$$|z^*| = |z| \text{ and } \arg(z^*) = -\arg(z)$$

$$\therefore f(z^*) = \ln(|z^*|) + i \arg(z^*)$$

$$= \ln(|z|) - i \arg(z)$$

$$\therefore (f(z))^* = f(z^*)$$

$$\mathbf{c} \quad f(z) = f(z^*)$$

$$\ln(|z|) + i \arg(z) = \ln(|z|) - i \arg(z)$$

$$2i \arg(z) = 0$$

$$\therefore \arg(z) = 0$$

$$\therefore z \text{ is a real number} \quad (\text{QED})$$

$$\mathbf{d} \quad \mathbf{i} \quad \ln(|z|) = 0 \therefore |z| = 1 \text{ and } \arg(z) \neq 0$$

$$z = x + iy \text{ where } x^2 + y^2 = 1$$

$$(x \neq 1 \text{ since } \arg(z) \neq 0)$$

$$\mathbf{ii} \quad \arg(z) = 0 \text{ and } \ln(|z|) < 0$$

$$y = 0, x > 0 \text{ and } |z| < 1$$

$$\therefore z = x \quad (0 < x < 1)$$

$$\mathbf{iii} \quad f(z) = 0 \Rightarrow |z| = 1 \text{ and } \arg(z) = 0$$

$$\therefore z = 1$$



Review exercise

$$\mathbf{1} \quad \mathbf{a} \quad 2 \operatorname{cis} \frac{4\pi}{3} = 2 \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) = -1 - \sqrt{3}i$$

$$\mathbf{b} \quad \sqrt{2} \operatorname{cis} 135^\circ = \sqrt{2} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = -1 + i$$

$$\mathbf{c} \quad \frac{2}{3 \operatorname{cis} \frac{5\pi}{6}} \times \frac{\operatorname{cis} \left(-\frac{5\pi}{6} \right)}{\operatorname{cis} \left(-\frac{5\pi}{6} \right)} = \frac{2}{3} \operatorname{cis} \left(-\frac{5\pi}{6} \right)$$

$$= \frac{2}{3} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = -\frac{\sqrt{3}}{3} - \frac{1}{3}i$$

$$\mathbf{2} \quad \mathbf{a} \quad 5 - 5i = 5\sqrt{2} \operatorname{cis} \frac{7\pi}{4}$$

$$\mathbf{b} \quad \frac{2}{1 + \sqrt{3}i} = \frac{2 \operatorname{cis} 0}{2 \operatorname{cis} \frac{\pi}{3}} = \operatorname{cis} \left(-\frac{\pi}{3} \right) \text{ or } \operatorname{cis} \frac{5\pi}{3}$$

$$\mathbf{c} \quad \frac{1-i}{\sqrt{3}-i} = \frac{\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right)}{2 \operatorname{cis} \left(\frac{11\pi}{6} \right)} = \frac{\sqrt{2}}{2} \operatorname{cis} \left(-\frac{\pi}{2} \right) \text{ or } \frac{\sqrt{2}}{2} \operatorname{cis} \left(\frac{11\pi}{12} \right)$$

$$3 \quad z = 2 \operatorname{cis} \frac{2\pi}{3} \quad w = 4 \operatorname{cis} \frac{5\pi}{4}$$

$$a \quad zw = 8 \operatorname{cis} \frac{23\pi}{12}, \quad \frac{z}{w} = \frac{1}{2} \operatorname{cis} \left(-\frac{7\pi}{12} \right) \text{ or } \frac{1}{2} \operatorname{cis} \frac{17\pi}{12}$$

$$z^2 w^3 = 4 \operatorname{cis} \frac{4\pi}{3} \times 64 \operatorname{cis} \frac{15\pi}{4} = 256 \operatorname{cis} \frac{13\pi}{12}$$

$$b \quad z = 2 \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) = -1 + i\sqrt{3}$$

$$w = 4 \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right) = -2\sqrt{2} - 2\sqrt{2}i$$

$$z + w = (-1 - 2\sqrt{2}) + i(\sqrt{3} - 2\sqrt{2})$$

$$z - w = (2\sqrt{2} - 1) + i(\sqrt{3} + 2\sqrt{2})$$

$$z^2 = 1 - 2i\sqrt{3} - 3 = -2 - 2\sqrt{3}i$$

$$\therefore \frac{z^2}{w} = \frac{-2 - 2\sqrt{3}i}{-2\sqrt{2} - 2\sqrt{2}i} \times \frac{-2\sqrt{2} + 2\sqrt{2}i}{-2\sqrt{2} + 2\sqrt{2}i}$$

$$= \frac{4\sqrt{2} - 4\sqrt{2}i + 4\sqrt{6}i + 4\sqrt{6}}{16}$$

$$\frac{z^2}{w} = \frac{(\sqrt{2} + \sqrt{6})}{4} + \frac{(\sqrt{6} - \sqrt{2})i}{4}$$

$$4 \quad z = a + i$$

$$a \quad \arg(z) = \frac{\pi}{3} \therefore \frac{1}{a} = \sqrt{3} \therefore a = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$b \quad z^2 = a^2 - 1 + 2ai \quad z^2 \text{ is real} \Rightarrow a = 0$$

$$c \quad |z - 1| = |z - 2i|$$

$$|(a - 1) + i| = |a - i|$$

$$(a - 1)^2 + 1 = a^2 + 1 \therefore a^2 - 2a + 1 = a^2$$

$$\therefore a = \frac{1}{2}$$

$$5 \quad z^5 = z \Rightarrow z = 0 \text{ or } z^4 = 1$$

$$z = 0 \text{ or } z^4 = \operatorname{cis} 0$$

$$z = \operatorname{cis} \frac{2k\pi}{4} \quad k = 0, 1, 2, 3$$

$$z = 0, \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{2}, \operatorname{cis} \pi, \operatorname{cis} \frac{3\pi}{2}$$

$$z = 0, 1, i, -1, -i$$

$$6 \quad z = \frac{1}{1 + i \tan \theta} \times \frac{1 - i \tan \theta}{1 - i \tan \theta} = \frac{1 - i \tan \theta}{\sec^2 \theta}$$

$$z = \cos^2 \theta i \sin \theta \cos \theta$$

$$|z|^2 = \cos^4 \theta + \sin^2 \theta \cos^2 \theta$$

$$= \cos^2 \theta (\cos^2 \theta + \sin^2 \theta)$$

$$|z|^2 = \cos^2 \theta \therefore |z| = \cos \theta$$

$$\arg(z) = \tan^{-1} \left(-\frac{\sin \theta}{\cos \theta} \right) = \tan^{-1}(\tan(-\theta)) = -\theta$$

$$\therefore z = \cos \theta \operatorname{cis} (-\theta)$$

$$7 \quad a \quad \frac{1-i}{4} = \frac{\sqrt{2}}{4} \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

$$\arg(a + ai) = \frac{\pi}{4} \quad (a > 0) \text{ or } -\frac{3\pi}{4} \quad (a < 0)$$

since $\frac{1-i}{4}$ and $a + ai$ are consecutive n th roots of z and since their arguments differ by $\frac{\pi}{2}$,

$$n = 4 \text{ and therefore } a = \frac{1}{4} \text{ or } -\frac{1}{4}$$

b The remaining n th roots of z are therefore

$$\frac{1+i}{4}, \frac{-1+i}{4}, \frac{-1-i}{4}$$

$$8 \quad (x + y)(x + wy)(x + w^2 y) = (x + y)(x^2 + w^2 xy + wxy + w^3 y^2)$$

$$= x^3 + w^2 x^2 y + wx^2 y + w^3 xy^2 + x^2 y + w^2 xy^2 + w^3 y^3$$

$$= x^3 + x^2 y(w^2 + w + 1) + xy^2(w^3 + w^2 + w) + w^3 y^3$$

$$w^3 = 1 \quad w^3 - 1 = 0$$

$$(w - 1)(w^2 + w + 1) = 0$$

$$w \neq 1 \therefore w^2 + w + 1 = 0$$

$$(x + y)(x + wy)(x + w^2 y) = x^3 + x^2 y(0) + xy^2(0) + y^3$$

$$= x^3 + y^3 \quad (\text{QED})$$

$$9 \quad z = -\sqrt{3} - i$$

$$a \quad |z| = 2 \arg(z) = 210^\circ$$

$$b \quad z = 2 \sin 210^\circ \sqrt[3]{z} = 2^{\frac{1}{3}} \operatorname{cis} 70^\circ \text{ or } 0.431 + 1.18i$$

$$c \quad z^n = 2^n \operatorname{cis} 210^\circ n$$

z^n is a positive real number

$$\therefore \cos 210^\circ n > 0 \text{ and } \sin 210^\circ n = 0$$

$$210^\circ n = 180^\circ k \quad k \in \mathbb{Z}$$

$$n = \frac{6}{7}k \quad k \in \mathbb{Z}$$

n is an integer $\therefore n = 6, 12, 18, \dots$

$$\cos 210^\circ n > 0 \therefore \text{smallest positive integer } n = 12$$

$$10 \quad (\cos \alpha + i \sin \alpha)^4 = \cos^4 \alpha + 4 \cos^3 \alpha i \sin \alpha - 6 \cos^2 \alpha \sin^2 \alpha - 4 i \cos \alpha \sin^3 \alpha + \sin^4 \alpha$$

$$\therefore \cos^4 \alpha + i \sin^4 \alpha = \cos^4 \alpha + 4i \cos^3 \alpha \sin \alpha - 6 \cos^2 \alpha \sin^2 \alpha - 4i \cos \alpha \sin^3 \alpha + \sin^4 \alpha$$

$$\therefore \cos^4 \alpha = \cos^4 \alpha - 6 \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha$$

$$\sin 4\alpha = 4 \cos^3 \alpha \sin \alpha - 4 \cos \alpha \sin^3 \alpha$$

$$\tan 4\alpha = \frac{\sin 4\alpha}{\cos 4\alpha} = \frac{4 \cos^3 \alpha \sin \alpha - 4 \cos \alpha \sin^3 \alpha}{\cos^4 \alpha - 6 \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha}$$

Dividing top and bottom by $\cos^4 \alpha$,

$$\tan^4 \alpha = \frac{4 \tan \alpha - 4 \tan^3 \alpha}{1 - 6 \tan^2 \alpha + \tan^4 \alpha} \quad (\text{QED})$$

$$12 \quad z + \frac{1}{z} = -1$$

$$a \quad \left(z + \frac{1}{z} \right)^2 = z^2 + 2 + \frac{1}{z^2} = 1$$

$$\therefore z^2 + \frac{1}{z^2} = -1$$

$$b \quad \left(z + \frac{1}{z} \right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$-1 = z^3 + \frac{1}{z^3} + 3 \left(z + \frac{1}{z} \right)$$

$$\therefore z^3 + \frac{1}{z^3} = 2$$

$$c \quad \left(z + \frac{1}{z} \right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$-1 = z^5 + \frac{1}{z^5} + 5 \left(z^3 + \frac{1}{z^3} \right) + 10 \left(z + \frac{1}{z} \right)$$

$$-1 = z^5 + \frac{1}{z^5} + 10 - 10 \therefore z^5 + \frac{1}{z^5} = -11$$

$$\begin{aligned} \mathbf{13\ a} \quad (x-z)(x-z^*) &= x^2 - x(z+z^*) + zz^* \\ z+z^* &= 2\operatorname{Re}(z) \quad zz^* = |z|^2 \\ \therefore (x-z)(x-z^*) &= x^2 - 2\operatorname{Re}(z)x + |z|^2 \quad (\text{QED}) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad z^8 = 1 &= \operatorname{cis} 0 \Rightarrow z = \frac{\operatorname{cis} 2k\pi}{8} \quad k = 0, 1, 2, \dots, 7 \\ z &= \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{4}, \operatorname{cis} \frac{\pi}{2}, \operatorname{cis} \frac{3\pi}{4}, \operatorname{cis} \pi, \operatorname{cis} \frac{5\pi}{4}, \\ &\quad \operatorname{cis} \frac{3\pi}{2}, \operatorname{cis} \frac{7\pi}{4} \\ &= 1, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, i, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \\ &\quad -i, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad x^8 - 1 &= (x-1)(x+1)(x-i)(x+i) \\ &= (x - (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}))(x - (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})) \\ &= (x - (-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}))(x - (-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})) \\ x^8 - 1 &= (x-1)(x+1)(x^2+1)(x^2-\sqrt{2}x+1) \\ &\quad (x^2+\sqrt{2}x+1) \end{aligned}$$

$$\mathbf{14} \quad w = \operatorname{cis} \alpha \quad 0 < \alpha < \frac{\pi}{2}$$

$$\begin{aligned} \mathbf{a} \quad 1+w &= (1+\cos \alpha) + i \sin \alpha \\ |1+w|^2 &= (1+\cos \alpha)^2 + \sin^2 \alpha \\ &= 1 + 2\cos \alpha + \cos^2 \alpha + \sin^2 \alpha \\ &= 2 + 2\cos \alpha \\ &= 2 + 2(2\cos^2 \frac{\alpha}{2} - 1) \end{aligned}$$

$$\begin{aligned} |1+w|^2 &= 4\cos^2 \frac{\alpha}{2} \\ \therefore |1+w| &= 2\cos \frac{\alpha}{2} \quad (\text{QED}) \end{aligned}$$

$$\begin{aligned} \arg(1+w) &= \arctan \left(\frac{\sin \alpha}{1+\cos \alpha} \right) \\ &= \arctan \left(\frac{2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{1 + (2\cos^2 \frac{\alpha}{2} - 1)} \right) \\ &= \arctan \left(\tan \frac{\alpha}{2} \right) \end{aligned}$$

$$\therefore \arg(1+w) = \frac{\alpha}{2} \quad (\text{QED})$$

$$\begin{aligned} \mathbf{b} \quad (1+w)^n &= (2\cos \frac{\alpha}{2} \operatorname{cis} \frac{\alpha}{2})^n \\ (2\cos \frac{\alpha}{2})^n \cos \frac{n\alpha}{2} &= \operatorname{Re}[(1+w)^n] \\ &= \operatorname{Re} \left[\sum_{k=0}^n \binom{n}{k} w^k \right] \\ &= \operatorname{Re} \left[\sum_{k=0}^n \binom{n}{k} (\cos k\alpha + i \sin k\alpha) \right] \\ &= \sum_{k=0}^n \binom{n}{k} \cos(k\alpha) \\ \therefore \sum_{k=0}^n \binom{n}{k} \cos(k\alpha) &\equiv (2\cos \frac{\alpha}{2})^n \cos(\frac{n\alpha}{2}) \quad (\text{QED}) \end{aligned}$$



Review exercise

1 Using question 11, if z is a zero then z^* is also a zero.

Therefore the 4 roots are $3 \pm i$ and $\sqrt{2} \operatorname{cis} \left(\pm \frac{\pi}{4} \right)$

$\therefore$ the polynomial is

$$\begin{aligned} p(z) &= (z-3+i)(z-3-i)(z-\sqrt{2} \operatorname{cis} \frac{\pi}{4}) \\ &\quad (z-\sqrt{2} \operatorname{cis} (-\frac{\pi}{4})) \\ &= [(z-3)^2 - i^2] [z-1-i] [z-1+i] \\ &= [(z-3)^2 - i^2] [(z-1)^2 - i^2] \\ &= [z^2 - 6z + 9 + 1] (z^2 - 2z + 1 + 1) \\ &= (z^2 - 6z + 10) (z^2 - 2z + 2) \\ &= z^4 - 8z^3 + 24z^2 - 32z + 20 \end{aligned}$$

Therefore $a = -8$, $b = 24$, $c = -32$, $d = 20$